CAAM 553
Exam 2
Due Dec. 13

Instructions:

• Answer the questions thoroughly; show your work to receive partial credit. However, please strive for clear and concise answers.

• You may not consult any books, notes, MATLAB, etc. You may use other parts of a question without proof, however.

• There is a four hour (uninterrupted) time limit. Do not look at the exam until you begin.

• Please start each problem on a new page and number your pages. Staple to this cover page to the front of your exam.

• Print your name on the line below

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• Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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• Write the time period for this exam:

    Time start: _______________    Time end: _______________
(1) (25 points) Let $A$ be a real $50 \times 5$ matrix and suppose that the Singular Value Decomposition (compact or “thin” SVD) of $A$ is

$$A = U S V^T$$

where $S = \text{diag}(5, 4, 3, 2, 1)$ and $U, V \in \mathbb{R}^{50 \times 5}$ have orthonormal columns $U_1, U_2, ..., U_5$ and $V_1, V_2, ..., V_5$ respectively. Justify your answers.

a. (5 points) What are the values of $\|A\|_2$ and $\kappa_2(A)$ (2-norm and 2-norm condition number)?

b. (5 points) Let $B = U_2 S_2 V_2^T$, where $U_2 = (U_1 \ U_2)$, $V_2 = (V_1 \ V_2)$ and $S_2 = \text{diag}(5, 4)$. What is the value of $\|A - B\|_2$?

c. (5 points) What is the numerical value of

$$\min\{\|A - C\|_2 : C \in \mathbb{R}^{50 \times 5}, \text{Rank}(C) = 2\},$$

and what matrix $C$ achieves this minimum value?

d. (5 points) Suppose the last two diagonal entries of $S$ were 0 so that $S = \text{diag}(5, 4, 3, 0, 0)$. Specify the rank of $A$ and give orthonormal bases for the range and null spaces of $A$.

e. (5 points) Suppose again that $S = \text{diag}(5, 4, 3, 0, 0)$. Use the SVD to construct an orthogonal projector onto the range of $A$ and an orthogonal projector onto the null space of $A^T$.

(2) (25 points)

a. (5 points) Let $U$ be a unitary matrix with eigenvalues $\lambda_j$. Show that $|\lambda_j| = 1$.

b. (5 points) Assume that $A = A^*$. The Rayleigh quotient of $A$ is defined as the quantity $\frac{x^* Ax}{x^* x}$.

Show that

$$\lambda_{\text{min}} \leq \frac{x^* Ax}{x^* x} \leq \lambda_{\text{max}}, \quad \forall x \in \mathbb{R}^n.$$  

b. (7 points) Let $H \in \mathbb{R}^{n \times n}$ be the Householder transformation $H = I - 2 \frac{vv^*}{v^* v}$. The eigenvalues of $H$ are $\pm 1$. How many eigenvalues are 1, and how many are $-1$? Characterize the corresponding eigenvectors.

c. (8 points) Let $A$ be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and associated eigenvectors $v_1, \ldots, v_n$. Assume that $\mu \neq \lambda_i$. Show that

$$\| (A - \mu I)^{-1} \|_2 = \max_{1 \leq i \leq n} \frac{1}{|\lambda_i - \mu|}.$$  

Explain all steps clearly.
(3) (25 points) Consider the first order initial value problem for \( \mathbf{u}(t) \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n} \)
\[
\mathbf{u}'(t) = -A \mathbf{u}(t), \quad \mathbf{u}(0) = \mathbf{u}_0.
\]
a. (10 points) Prove that the trapezoid rule
\[
\frac{x_{k+1} - x_k}{h} = \frac{1}{2}(f_k + f_{k+1}).
\]
is convergent and second order accurate for a scalar ODE.
b. (5 points) Show that, if \( A = -A^* \), \( x^* A x = 0 \) for all \( x \in \mathbb{R}^n \).
c. (5 points) Show that, if \( A = -A^* \), all eigenvalues of \( A \) are purely imaginary.
d. (5 points) One can show from the ODE that if \( A = -A^* \), then the solution \( \mathbf{u}(t) \) satisfies the property
\[
\frac{d}{dt}\|\mathbf{u}(t)\|_2^2 = 0.
\]
It can be useful to ensure this property holds under a time-stepping scheme as well. Suppose that this ODE is solved using the trapezoid rule, with
\[
\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{h} = \frac{1}{2}(f_{k+1} + f_k) = -A \left( \frac{\mathbf{u}^{k+1} + \mathbf{u}^k}{2} \right).
\]
Show that if \( A = -A^* \), the norm of the solution is conserved, such that
\[
\|\mathbf{u}^{k+1}\|_2^2 = \|\mathbf{u}^k\|_2^2.
\]

(4) (25 points)
a. (15 points) Consider the backwards difference formula (BDF)
\[
3x_{k+2} - 4x_{k+1} + x_k = 2hf_{k+2}.
\]
Show that this method is convergent, and determine the order.
b. (10 points) Consider the application of this method to the ODE \( x'(t) = \lambda x(t) \) with step size \( h \). Determine whether the method is stable for \( \lambda h = -1/2 \). You may show this using the stability polynomial or through direct computation. What happens when \( \lambda h = 3/2 \)?