(1) (40 points) **This problem is pledged!** You may not discuss this with anyone but your instructor. You may not consult any source other than Prof. Embree’s lecture notes, your class notes or the textbook to help you with the problem.

Let \( A \in \mathbb{C}^{m \times m} \) be nonsingular.

i. (20 points) Show that \( A \) has an LU factorization if and only if for each \( k \) with \( 1 \leq k \leq m \), the upper-left \( k \times k \) block \( A(1 : k, 1 : k) \) is nonsingular. (Hint: prove that the row operations of Gaussian elimination leave the determinants \( det(A(1 : k, 1 : k)) \) unchanged.)

ii. (20 points) Prove that this LU factorization is unique.

(2) (30 points) Assume \( A \in \mathbb{C}^{m \times m} \) is nonsingular. Suppose \( A \) is written in the block form

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

where \( A_{11} \) is \( n \times n \) and \( A_{22} \) is \( (m - n) \times (m - n) \).

i. (10 points) Verify the formula

\[
\begin{bmatrix}
I & 0 \\
-A_{21}A_{11}^{-1} & I
\end{bmatrix} \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22} - A_{21}A_{11}^{-1}A_{12}
\end{bmatrix}
\]

for “elimination” of the block \( A_{21} \). The matrix \( A_{22} - A_{21}A_{11}^{-1}A_{12} \) is known as the **Schur complement** of \( A_{11} \) in \( A \).

ii. (20 points) Suppose \( A_{21} \) is eliminated row by row by means of \( n \) steps of Gaussian elimination. Show that the bottom-right \( (m - n) \times (m - n) \) block of the result is again \( A_{22} - A_{21}A_{11}^{-1}A_{12} \).
i. Implement the power method for finding the dominant eigenvalue $\lambda_1$ |$\lambda_1$| > |$\lambda_2$| ≥ ... ≥ |$\lambda_n$|. Use the power method to find the dominant eigenvalue and corresponding eigenvector for the Hilbert matrix with entries

$$A_{i,j} = \frac{1}{i + j + 1}$$

for $i, j = 1, \ldots, n$. (Note the dominant eigenvalue is well-separated in this case.)

Test for $n = 4 : 4 : 20$. How many iterations are needed?

ii. Use the power method (with the necessary modifications) to find the smallest eigenvalue of the matrix with $n = 16$. How accurate is the eigenvalue?

iii. Is the error estimate in ii. consistent with the estimate $\min_{\lambda \in \sigma(A)} |\lambda - \mu| \leq \|E\|_2$ where $\mu$ is the eigenvalue of the perturbed matrix $A + E$?

iv. Construct a numerical example where the power method does not work.

Thm: (Bauer-Fiske) If $\mu$ is an eigenvalue of $A + E \in \mathbb{C}^{n \times n}$ and $A$ is diagonalizable, i.e.

$$A = P^{-1}AP = D = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

then $\min_{\lambda \in \sigma(A)} |\lambda - \mu| \leq \|P^{-1}\|_p \|P\|_p \|E\|_p$ for any $p$-norm.

iv. Construct a numerical example where the power method does not work.

(4) (25 points) This problem is pledged! You may not discuss this with anyone but your instructor. You may not consult any source other than NA, NLA, Prof. Embree’s lecture notes or your in-class notes to help you with the problem.

Suppose that $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite. Consider the following iteration:

$$A_0 = A$$

for $k = 1, 2, 3, \ldots$

$$A_{k-1} = G_k G_k^T$$

Here $G_k G_k^T$ is the Cholesky factorization of a symmetric positive definite matrix.

i. Show that this iteration is well defined, i.e. $G_k G_k^T$ is symmetric and positive definite.

ii. Show that if

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

with $a \geq c$, $A$ has eigenvalues $\lambda_1 \geq \lambda_2 > 0$ and $A_k$ converges to $\text{diag}(\lambda_1, \lambda_2)$. 