CAAM 553
Homework 12 Solutions

(1) (25 points) The linear system

\[ y' = Ay, \quad y(0) = y_0 \]  

where A is symmetric is solved by the explicit Euler’s method. Let \( e_n = y_n - y(nh) \), \( n = 0, 1, \ldots \) and prove that

\[ \|e_n\|_2 \leq \|y_0\|_2 \max_{\lambda \in \sigma(A)} |(1 + \lambda h)^n - e^{hn\lambda}|, \]

where \( \sigma(A) \) is the set of eigenvalues of A. **Solution:** Since it is symmetric, A is unitarily diagonalizable as \( V\Lambda V^* \). The exact solution is given by the matrix exponential \( y(t) = e^{At}y_0 = V e^{\Lambda t}V^*y_0 \), while the Euler solution is

\[ y_{k+1} = (I + hA)y_k = (1 + hA)^{k+1}y_0 = P(1 + h\Lambda)^{k+1}P^*y_0, \]

since \( I + hA \) has the same eigenvectors as A. Subtracting this from the exact solution at \( t_k = hk \)

\[ \|y(hk) - y_k\| = \|e_n\| \leq \|P\| \|(1 + h\Lambda)^k - e^{\Lambda hk}\||P\|\|y_0\|. \]

The quantity \( \|(1 + h\Lambda)^k - e^{\Lambda hk}\| \) is the largest eigenvalue, and since both matrices are diagonal, this is equal to

\[ \|(1 + h\Lambda)^k - e^{\Lambda hk}\| = \max_{\lambda \in \sigma(A)} ((1 + h\lambda)^k - e^{\lambda hk}). \]

(2) (25 points) Implement implicit Euler method to solve the initial value problem

\[ y' = f(x, y) \]

where \( y = (y_1, y_2), f(x, y) = (f_1(x, y), f_2(x, y)) \) and \( y(0) = y_0 \). Use repeated Richardson extrapolation to improve the results.

Solve

\[ x^2y'' + xy' + x^2y = 0, \]

with initial condition \( y(0) = 1 \) and \( y'(0) = 0 \). Construct the first order system, solve on \( x \in [0, 3\pi] \) and use repeated Richardson extrapolation to compute \( y(3\pi) \). (The exact solution to the problem is \( J_0(x) \), the Bessel function of the first kind.)

**Solution:** The second order equation can be recast as the following first order system

\[ y' = w \]

\[ w' = -xw - x^2y \]

The system can be solved at each step of implicit Euler using Newton or a fixed point iteration. Alternatively, the system admits an analytic solution which can also be used.

See code `HW12_prob2.m` for the implementation. Note that the final time must be fixed, so we double the number of timesteps taken at each Richardson step. The timestep is then determined by dividing final time by number of steps.