CAAM 553
Homework 2
Due Sept. 13

(1) Let $x_0, x_1$ be two successive points from a secant method applied to solving $f(x) = 0$ with $f_0 = f(x_0), f_1 = f(x_1)$. Show that regardless of which point $x_0$ or $x_1$ is regarded as the most recent point, the new point derived from the secant step will be the same.

(2) Which of the following iterations will converge to the indicated fixed point $x_*$ (provided $x_0$ is sufficiently close to $x_*$)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.
   i. (5 pts) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, \ x_* = 2$
   ii. (5 pts) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, \ x_* = 3^{1/3}$
   iii. (5 pts) $x_{n+1} = \frac{12}{1+x_n}, \ x_* = 3$

(3) This problem is pledged! You may not discuss this with anyone but your instructor. You may not consult any source other than NA, NLA, Prof. Embree’s lecture notes or your in-class notes to help you with the problem.
   Consider applying Newton’s method to a real cubic polynomial.
   i. (10 pts) In the case that the polynomial has three distinct real roots, $x = \alpha, x = \beta$ and $x = \gamma$, show that the starting guess $x_0 = \frac{1}{2}(\alpha + \beta)$ will yield the root $\gamma$ in one step.
   ii. (10 pts) Give a heuristic (e.g. geometric) argument showing that if two roots coincide (say $\beta = \gamma$), there is precisely one starting guess $x_0$ (other than the double root) for which Newton will fail, and that this one separates the basins of attraction for the distinct roots.
   iii. (10 pts) Extend the argument in part ii. to the case when all three roots again are distinct. Explain why there are now infinitely many starting guesses $x_0$ for which the iteration will fail.

(4) This problem is pledged! You may not discuss this with anyone but your instructor. You may not consult any source other than NA, NLA, Prof. Embree’s lecture notes or your in-class notes to help you with the problem.
   The following data are taken from a polynomial of degree $\leq 5$. What is the degree of the lowest degree polynomial which can reproduce this data?

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>-5</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>
(5) We want to construct a rational function of the form

\[ R(z) = \frac{\alpha + \beta z}{1 + \gamma z} \]

that interpolates the data \((z_1, f_1), (z_2, f_2), (z_3, f_3)\) at distinct points \(z_1, z_2\) and \(z_3\). In other words, we seek \(\alpha, \beta,\) and \(\gamma\) such that

\[ R(z_j) = f_j, \quad j = 1, 2, 3. \]

Show how you can determine \(\alpha, \beta,\) and \(\gamma\) by setting a linear system \(Ax = b\) for the unknown vector \(x = (\alpha, \beta, \gamma)^T\). (Just write the system, you do not need to solve it).

(6) We studied in class interpolation of functions defined in 1D. We can adapt the technique to higher dimensions. For instance, let

\[ f(x, y) = e^x \sin y \]

We want to construct a polynomial of the form

\[ p(x, y) = c_0 + c_1 x + c_2 y + c_3 xy + c_4 x^2 + c_5 y^2 \]

that interpolates \(f\) at the points \((x_i, y_i)\):

\[ p(x_i, y_i) = f(x_i, y_i); \quad 0 \leq i \leq 5. \]

i. (5 points) Set up a linear system \(Ac = f\) to determine the coefficients \(c_0, \ldots, c_5\).

ii. (5 points) Write a Matlab code to determine \(c\) when the data points are

\((0, 0), (0, 2), (1, 0), (1, 2), (2, 1), (2, 3)\)

Report your value for \(c\).

iii. (10 points) Plot your polynomial \(p\) over \(x \in [-1, 3], y \in [-1, 3]\) using Matlab’s surf command. Compare this plot to the similar plot for \(f\) which can be obtained in the following manner.

\[ f=\text{inline}('e^x.*\sin(y)','x','y'); \]
\[ [xx,yy]=\text{meshgrid}(\text{linspace}(-1,3,25),\text{linspace}(-1,3,25)); \]
\[ zz=f(xx,yy); \]
\[ \text{figure(1)}, \text{clf} \]
\[ \text{surf}(xx,yy,zz) \]

Remark: include the paper copy of your Matlab code with your homework.