(1) Divided differences and verifying error bounds.

i. Given \( x_0 = -0.2, x_1 = 0 \) and \( x_2 = 0.2 \) construct a second degree polynomial to approximate \( f(x) = e^x \) via Newton’s divided differences.

**Soln:**

\[
x_0 = -0.2 \quad f(x_0) = e^{-0.2} \\
x_1 = 0 \quad f(x_1) = 1 \\
x_2 = 0.2 \quad f(x_2) = e^{0.2}
\]

Reading off the coefficients, we find the polynomial is

\[p_2(x) = e^{-0.2} + \frac{e^{-0.2} - 1}{-0.2} (x + 0.2) + \frac{e^{-0.2} + e^{0.2} - 2}{0.08} (x + 0.2)x.\]

ii. Derive a pointwise and a global error bound for \( p_2(x) \) when \( x \in [-0.2, 0.2] \).

**Soln:** In class we found

\[|E(t)| = |f(t) - p_2(t)| \leq \max_{\alpha \in [t,x_0,\ldots,x_2]} \frac{f^{(3)}(\alpha)}{3!} |(t-x_0)(t-x_1)(t-x_2)|.\]

Applying this bound to our function we find

\[|E(t)| \leq \frac{e^{0.2}}{3!} |(t+0.2)(t-0.2)t|.\]

Alternatively we can find a bound for all \( x \in [-0.2, 0.2] \). To do this we need to find the maximum of the polynomial \( q(x) = (x+0.2)(x-0.2)x \) in the interval. Note that \( q(x) \) is zero at both end points. We will check the derivative to find \( q'(x) = -0.04 + 3x^2 \) which is zero at \( x = \pm \sqrt{\frac{0.04}{3}} \). Since \( |q(x)| \) is symmetric in \([-0.2, 0.2]\), we can choose either point. So the maximum \( q \) can be is \( 0.0030792 \). Thus the bound for the error is \(|E| \leq 6.2682e - 04\).

iii. Compute the error \( E(0.1) = f(0.1) - p_2(0.1) \). How does this compare with the error bound?

**Soln:** \(|E(0.1)| = |e^{0.1} - p_2(0.1)| = 5.1377e - 04\) The first error bound says \(|E(0.1)| \leq 6.1070e - 04\) which is true. Notice the global error bound is less accurate than the first one.

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i. Show there is a unique cubic polynomial \( p(x) \) for which

\[
p(x_0) = f(x_0) \quad p(x_2) = f(x_2) \\
p'(x_1) = f'(x_1) \quad p''(x_1) = f''(x_1),
\]

where \( f(x) \) is a given function and \( x_0 \neq x_2 \). Derive a formula \( p(x) \).

**Soln:** First, we will prove that the polynomial is unique. Suppose there exist two polynomials \( p(x) \) and \( q(x) \). Let \( w(x) = p(x) - q(x) \). Our goal is to show \( w(x) = 0 \).
We know \( w(x_0) = 0 \) and \( w(x_2) = 0 \). This means there exist an \( \alpha \) between \( x_0 \) and \( x_2 \) such that \( w'(\alpha) = 0 \).

We also know that \( w'(x_1) = 0 \) and \( w''(x_1) = 0 \). This means that \( x_1 \) is a double root of \( w'(x) \). Now \( w(x) \in \mathcal{P}_2 \). If \( \alpha \neq x_1 \), then \( w'(x) = 0 \) and \( w(x) = 0 \).

Now, we analyze \( w(x) \). Since \( w''(x_1) = 0 \), \( x_1 \) is a point of an inflection. Therefore there is a point \( \alpha \neq x_1 \) such that \( w'(\alpha) = 0 \); otherwise there is no way for \( w(x_0) = w(x_2) = 0 \).

Now we will derive the formula of \( p(x) \). We set \( p(x) = f(x_0)l_0(x) + f'(x_1)l_1(x) + f''(x_1)l_2(x) + f(x_2)l_3(x) \).

We now have to find \( l_j(x) \).

For \( l_0(x) \), we want \( l_0(x_0) = 1 \), \( l_0(x_2) = 0 \), \( l_0(x_1) = 0 \), \( l_0'(x_1) = 0 \), and \( l_0''(x_1) = 0 \). This implies that \( x_1 \) is a double root of \( l_0'(x) \). Thus \( l_0(x) = (x - x_1)^2 \). Integrating we find \( l_0(x) = a(x - x_1)^3 + b \), where \( a \) and \( b \) are uniquely determined by the remaining two constraints. Upon doing the algebra, you find \( a = \frac{1}{(x_0 - x_1)^2 - (x_2 - x_1)^2} \) and \( b = \frac{-(x_2 - x_1)^2}{(x_0 - x_1)^2 - (x_2 - x_1)^2} \). Thus \( l_0(x) = \frac{(x - x_1)^3 - \frac{(x_2 - x_1)^3}{(x_0 - x_1)^2 - (x_2 - x_1)^2}}{(x_0 - x_1)^2 - (x_2 - x_1)^2} \).

The same reasoning applies to \( l_3(x) \). Thus \( l_3(x) = \frac{(x - x_1)^3 - (x_0 - x_1)^3}{(x_2 - x_1)^3 - (x_0 - x_1)^3} \).

We need \( l_1(x) \) to satisfy \( l_1(x_0) = 0 \), \( l_1'(x_1) = 1 \), \( l_1(x_2) = 0 \), and \( l_2'(x_1) = 0 \). The first and third facts tell us that \( l_1(x) = (x - x_0)(x - x_2)(ax + b) \). We will use the other two facts to identify \( a \) and \( b \).

\[
l_1'(x) = (x - x_0)(x - x_2)a + (ax + b)(x_0 + x - x_2)
= a(x - x_0)(x - x_2) + (ax + b)(2x - x(x_0 + x_2))
\]

\[
l_1''(x) = a(x - x_0) + a(x - x_2) + 2(ax + b) + a(x - x_0 + x - x_2)
= a(2x - (x_0 + x_2)) + 2ax + 2b + a(2x - (x_0 + x_2))
= a(6x - 2(x_0 + x_2)) + 2b
\]

We can now solve for \( a \) and \( b \).

\[
a = \frac{1}{(x_1 - x_0)(x_1 - x_2) + x_1(2x_1 - (x_0 + x_2)) - (2x_1 - (x_0 + x_2))(3x_1 - (x_0 + x_2))}
\]

and

\[
b = \frac{-(3x_1 - (x_0 + x_2))}{(x_1 - x_0)(x_1 - x_2) + x_1(2x_1 - (x_0 + x_2)) - (2x_1 - (x_0 + x_2))(3x_1 - (x_0 + x_2))}
\]

The same approach applies for \( l_2(x) \). The result is \( l_2(x) = (x - x_2)(x - x_0)(cx + d) \) where \( c = \frac{-2(x_1 - (x_0 + x_2))}{2((x_1 - x_2)(x_1 - x_0) + 2x_1 - (x_0 + x_2))} \) and \( d = 1/2 - a(3x_1 + (x_0 + x_2)) \).

ii. Let \( x_0 = -1 \), \( x_1 = 0 \) and \( x_2 = 1 \). Assuming \( f(x) \in C^4([-1, 1]) \), show that for \(-1 \leq x \leq 1 \),

\[
f(x) - p(x) = \frac{x^4 - 1}{4!} f^4(\eta_x)
\]

for some \( \eta_x \in [-1, 1] \).

**Sohn:** Let \( \Psi(x) = x^4 - 1 \) and \( E(x) = f(x) - p(x) \). Assume \( t \) is not a zero of \( \Psi \). Let \( G(x) = E(x) - \frac{\Psi(x)}{\Psi'(t)} E(t) \).

Note that \( t, x_0 \) and \( x_2 \) are zeros of \( G \). This means there is are two points \( \alpha \in (x_0, t) \) and \( \beta \in (t, x_2) \) such that \( G'(x) = 0 \). Without loss of generality, assume \( t < x_1 \) so we know that \( \alpha \neq x_1 \). We now consider two cases: First if is just \( x_1 \) a double root
of $G'(x)$, then there must be a $\beta \in (x_1, x_2)$ such that $G'(\beta) = 0$. Hence $G'(x)$ had four zeros counting $x_1$ twice. Second case: $x_1$ is a root of $G'(x)$ with multiplicity greater than 2. Then it is is possible that there is not another point between $x_1$ and $x_2$ where $G'(x)$ is zero. This is okay because $G'(x)$ still has at least 4 zeros (with $x_1$ having multiplicity greater than 2).

In any case, $G'(x)$ has at least 4 zeros. So $G''(x)$ has at least 3 zeros. $G^{(3)}(x)$ has at least two zeros and $G^{(4)}(x)$ has at least 1 zero. Let $\eta \in [x_0, x_2]$ denote a root of $G^{(4)}(x)$. This means

$$0 = G^{(4)}(\eta) = f^{(4)}(\eta) - \frac{4!}{\Psi(t)} E(t).$$

Solving for $E(t)$ we find the expression for the error.

$$E(t) = f(t) - p(t) = \frac{t^4 - 1}{4!} f^{(4)}(\eta)$$

(3) Let $f$ be a $C^3$ function on $[-h, h]$ where $h > 0$. Suppose $f$ has been sampled at the points $-h$ and $h$ and that $f(\pm h)$ are known, but only up to rounding errors $\epsilon_\pm$ respectively. Define

$$E(h) = \frac{(f(h) + \epsilon_+) - (f(-h) + \epsilon_-)}{2h} - f'(0).$$

i. Show that there exists $\eta \in (-h, h)$ such that

$$E(h) = \frac{1}{6} h^2 f'''(\eta) + \frac{\epsilon_+ - \epsilon_-}{2h}.$$ 

In other words, deduce that

$$|E(h)| \leq \frac{1}{6} h^2 M + \frac{\epsilon}{h}$$

where $M = \|f''''\|_{\infty}$ and $\epsilon = \max(\epsilon_+, \epsilon_-)$. Hint: Taylor’s theorem could be helpful.

Solv: Taylor’s theorem states that $f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \frac{x^3}{3!} f^{(3)}(\eta)$ for some $\eta$ in the interval containing 0 and $x$.

Thus

$$f(h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + \frac{h^3}{3!} f^{(3)}(\eta_1)$$

and

$$f(-h) = f(0) - f'(0)h + f''(0)\frac{h^2}{2!} - \frac{h^3}{3!} f^{(3)}(\eta_2)$$

where $\eta_1$ is in the interval between 0 and $h$ and $\eta_2$ is in the interval between 0 and $-h$.

Subtract the two equations

$$f(h) - f(-h) = 2hf'(0) + \frac{h^3}{6} \left(f^{(3)}(\eta_1) - f^{(3)}(\eta_2)\right).$$

We now find

$$\frac{f(h) - f(-h)}{2h} = f'(0) + \frac{h^2}{12} \left(f^{(3)}(\eta_1) - f^{(3)}(\eta_2)\right).$$

There exist an $\eta \in (-h, h)$ such that

$$\frac{1}{2} \left(f^{(3)}(\eta_1) - f^{(3)}(\eta_2)\right) = f^{(3)}(\eta).$$
We proved that
\[ E(h) = \frac{1}{6} h^2 f'''(\eta) + \frac{\epsilon_+ - \epsilon_-}{2h}. \]
From this we have
\[ |E(h)| \leq \frac{1}{6} h^2 M + \frac{|\epsilon_+| + |\epsilon_-|}{2h} \leq \frac{1}{6} h^2 M + \frac{\epsilon}{h}. \]

ii. Show further that the right-hand side of the last inequality achieves its minimum value when
\[ h = \left( \frac{3\epsilon}{M} \right)^{1/3}. \]

**Soln:** Consider the function
\[ \phi(h) = \frac{1}{6} h^2 M + \frac{\epsilon}{h}. \]
We are looking for the minimum where \( \phi'(h) = 0 \)
\[ \phi'(h) = \frac{1}{3} h M - \frac{\epsilon}{h^2} \]
for \( h = (3\epsilon/M)^{1/3}. \) To check that this indeed a minimum, we look at the second derivative.
\[ \phi''(h) = \frac{1}{3} M + \frac{2\epsilon}{h^3} > 0 \]
This means that the function is concave up. Thus this means that \( h = (3\epsilon/M)^{1/3} \) is the minimum.

(4) Verify that the Hermite basis functions \( A_i(x), B_i(x) \)
\[ A_i(x) = (1 - 2(x - x_i)\ell'_i(x))\ell_i(x)^2, \quad B_i(x) = (x - x_i)\ell_i(x)^2, \]
satisfy the properties
\[ A_i(x_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j \end{cases}, \quad B_i'(x_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j \end{cases}, \quad B_i(x_j) = 0. \]

**Soln:** For \( j \neq i, \ell_i(x_j) = 0 \) gives \( A_i(x_j) = (1 - 2(x_j - x_i)\ell'_i(x_i))\ell_i(x_j)^2 = 0. \) For \( i = j, \)
\( (x_j - x_i) = 0 \) and \( \ell_i(x_i) = 1 \) give \( A_i(x_i) = (1 - 2(x_i - x_i)\ell'_i(x_i))\ell_i(x_i)^2 = 1. \) For \( i \neq j, \)
\( A'_i(x_j) = 0 \) by the product rule and the fact that a factor of \( \ell_i(x_j) \) multiplies each term.
If \( i = j, \) then the product rule and \( \ell_i(x_i) = 1 \) gives
\[ A'_i(x_i) = (1 - 2(x_i - x_i)\ell'_i(x_i))2\ell_i(x_i)\ell'_i(x_i) + \ell_i(x_i)^2(-2\ell'_i(x_i)) = 0. \]
\( B_i(x_j) = 0 \) for \( i \neq j \) since \( \ell_i(x_j) = 0, \) and for \( i = j, \) \( B_i(x_i) = 0 \) by \( (x_j - x_i) = 0. \) For
\( B'_i(x), \) we have \( B'_i(x) = \ell_i(x)^2 + (x - x_i)2\ell'_i(x)\ell_i(x). \) The first term evaluates to 1 and the second term vanishes at \( x_i. \)

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Recall the Lagrange basis functions \( l_j(x) \) define for \( x_0, \ldots, x_n \) are defined by \( l_j(x) = \prod_{i \neq j} \frac{x - x_i}{x_j - x_i} \) for \( i = 0, \ldots, n. \)
i. Prove that for any \( n \geq 1 \)
\[ \sum_{j=0}^{n} l_j(x) = 1 \quad \text{for all } x \in \mathbb{R}. \]
**Soln:** Define $G(x) = \sum_{j=0}^{n} l_j(x) - 1$. Notice that $G(x_i) = 0$ for $i = 1, \ldots, n$ but $G$ is a polynomial of degree $n$. These two facts put together imply $G(x) = 0$.

ii. Define $\Psi(x) = (x-x_0) \cdots (x-x_n)$. Show that the polynomial interpolant of degree $n$ that interpolates the data $(x_i, f(x_i))$ for $i = 0, \ldots, n$ can be written in the form

$$p_n(x) = \sum_{i=0}^{n} \frac{\Psi(x)}{(x-x_i)\Psi'(x_i)} f(x_i)$$

provided $x \neq x_i$ for all $i$.

**Soln:** We know

$$p_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i).$$

So we need to show that $\frac{\Psi(x)}{(x-x_i)\Psi'(x_i)} = l_i(x)$.

First notice that

$$\frac{\Psi(x)}{(x-x_i)} = \prod_{j=0, j\neq i}^{n} (x-x_j).$$

The derivative of $\Psi$ is given by

$$\Psi'(x) = \sum_{k=0}^{n} \prod_{j=0,j\neq k}^{n} (x-x_j)$$

so $\Psi'(x_i) = \prod_{j=0,j\neq i}^{n} (x_i-x_j)$. Thus $l_i(x) = \frac{\Psi(x)}{(x-x_i)\Psi'(x_i)}$.

iii. Define next

$$w_i = 1 \Psi'(x_i).$$

Show that the polynomial interpolant of degree $n$ that interpolates the data $(x_i, f(x_i))$ for $i = 0, \ldots, n$ can be written in the form

$$p_n(x) = \sum_{i=0}^{n} \frac{w_if(x_i)}{x-x_i} \prod_{j=0}^{n} \frac{w_j}{x-x_j}.$$

provided $x \neq x_i$ for all $i$.

This form is called the barycentric representation of $p_n$ giving it as a weighted sum of the values $f(x_0), \ldots, f(x_n)$.

**Soln:** Notice that there is no $\Psi(x)$ in the formula. Thus we need to find an expression for it in terms of everything else.

From parts i. and ii., we know

$$\sum_{j=0}^{n} l_j(x) = \sum_{j=0}^{n} \frac{\Psi(x)}{(x-x_j)\Psi'(x_j)} = 1.$$ 

By plugging in the definition of $w_j$, we find

$$\sum_{j=0}^{n} w_j \Psi(x) = 1.$$ 

We now can find an expression for $\Psi(x)$.

$$\Psi(x) = \frac{1}{\sum_{j=0}^{n} \frac{w_j}{x-x_j}}$$
Plugging this into the formula we derived in part ii. we find

\[ p_n(x) = \frac{\sum_{j=0}^{n} \frac{w_j f(x_j)}{x-x_j}}{\sum_{j=0}^{n} \frac{w_j}{x-x_j}}. \]