(1) (20 points) Let \( t_j = j \ast h \) where \( h = 1/(n+1) \) and \( j = 1, 2, \ldots, n \). Note these are equally spaced points in the interval \((0, 1)\). Consider the first \( n \) basis functions from the following polynomial basis sets.

i. The monomial basis \( 1, t, t^2, \ldots, t^{n-1} \);

ii. The Lagrange basis \( \ell_j(t) = \prod_{k=1, k \neq j}^{n} \frac{t - t_k}{t_j - t_k} \) for \( 1 \leq j \leq n \);

iii. The Newton basis \( \phi_j(t) = \prod_{k=1}^{j-1} (t - t_k) \) for \( 1 \leq j \leq n \);

iv. The Chebyshev polynomials of the first kind using the 3-term recursion \( T_{k+1}(t) = 2tT_k(t) - T_{k-1}(t) \) with \( T_0(t) \equiv 1 \) and \( T_1(t) = t \).

For each of these cases, plot the graphs of the first 5 basis functions on the same graph, using different line types for each basis function. The first three (monomial, Lagrange, Newton) should be plotted for \( t \in [0, 1] \) and the last one (Chebyshev) should be plotted for \( t \in [-1, 1] \). There should be four graphs (each with title, axis labels, and legends).

What is different and/or similar about the sets of basis function?

(2) (35 points) **This problem is pledged!** You may not discuss this with anyone but your instructor. You may not consult any source other than NA, NLA, Prof. Embree’s lecture notes or your in-class notes to help you with the problem.

i. (10 points) Find at what values of \( x \) the relative maximum and minima of \( T_n(x) = \cos(n \cos^{-1} x) \) occur, for \( n = 0, 1, \ldots \) on \([-1, 1]\).

ii. (10 points) Find the zeros of \( T_n(x) \) on \([-1, 1]\).

iii. (15 points) Show that \( T_n \) are orthogonal with respect to the weight \( 1/\sqrt{1-x^2} \), and compute their weighted \( L^2 \) norms for \( n = 0, 1, 2, \ldots \).

(3) (15 points) Let \([a, b]\) be any fixed interval. Given any \( \epsilon > 0 \), show that there exists some \( f \in C([a, b]) \) such that
\[
\|f\|_{L^2} \leq \epsilon \quad \text{while} \quad \|f\|_{L^\infty} \geq 1/\epsilon.
\]
[Süli and Mayers, problem 8.1]

(4) (30 points) Consider approximations to \( x \) for \( x \in [0, 1] \).

i. (10 points) Find the line that best approximates \( \sqrt{x} \) in the minimax (\( L^\infty \)) sense, and report the error. Hint: Consider the oscillation theorem, and the derivative of the error at its extreme points.

ii. (10 points) Find the line that best approximates \( \sqrt{x} \) in the least-square (\( L^2 \)) sense, and report the error.

iii. (10 points) For a general interval \([a, b]\), prove that for all \( f \in C([a, b]) \),
\[
\min_{p \in P_n} \|f - p\|_{L^2} \leq \sqrt{b-a} \min_{p \in P_n} \|f - p\|_{L^\infty}.
\]

Confirm that your solutions to part i. and part ii. are consistent with this bound.