(1) (20 points) Let $t_j = j \ast h$ where $h = 1/(n+1)$ and $j = 1, 2, \ldots, n$. Note these are equally spaced points in the interval (0, 1). Consider the first $n$ basis functions from the following polynomial basis sets.

i. The monomial basis $1, t, t^2, \ldots, t^{n-1}$;

ii. The Lagrange basis $\ell_j(t) = \prod_{k=1, k \neq j}^{n} \frac{t-t_k}{t_j-t_k}$ for $1 \leq j \leq n$;

iii. The Newton basis $\phi_j(t) = \prod_{j-1}^{j} (t-t_k)$ for $1 \leq j \leq n$;

iv. The Chebyshev polynomials of the first kind using the 3-term recursion $T_{k+1}(t) = 2tT_k(t) - T_{k-1}(t)$ with $T_0(t) \equiv 1$ and $T_1(t) = t$.

For each of these cases, plot the graphs of the first 5 basis functions on the same graph, using different line types for each basis function. The first three (monomial, Lagrange, Newton) should be plotted for $t \in [0, 1]$ and the last one (Chebyshev) should be plotted for $t \in [-1, 1]$. There should be four graphs (each with title, axis labels, and legends).

What is different and/or similar about the sets of basis function?

Figure 0.1 illustrates the polynomial bases. Notice that the monomials and the Newton basis behave similarly. But the Newton basis does go all the way to 1 as $x$ goes to 1 and
for small $x$ there is little contribution from the higher order polynomials. This matches
our intuition from the definition the Newton interpolation.

For the Lagrange basis, notice that there is nice coverage through out the interval of
the range. It is interesting to note that $l_2$ is symmetric and $l_0$ is a mirror copy of $l_5$.
Similarly, $l_1$ is a mirror copy of $l_4$.

Like the Lagrange basis, the Chebyshev basis nicely covers the entire range throughout
the interval. Notice the oscillatory behavior of the functions and that the maximum in
absolute value is 1. The functions achieve that maximum for each interval between roots.

(2) (35 points) **This problem is pledged!** You may not discuss this with anyone but
your instructor. You may not consult any source other than NA, NLA, Prof. Embree’s
lecture notes or your in-class notes to help you with the problem.

i.(10 points) Find the relative maximum and minima of $T_n(x) = \cos(n \cos^{-1} x)$, $n = 0, 1, \ldots$ on $[-1, 1]$.

**Soln:** Rewrite $T_n = \cos(n \theta)$ where $\theta = \cos^{-1}(x)$. We know that the maximum
$|T_n| = 1$ which occurs when $n \theta = j \pi$ for $j = 0, \ldots, n$. Thus the relative maximum
and minima occur at $x = \cos \left( \frac{2\pi j}{n} \right)$ for $j = 0, \ldots, n$.

ii. (10 points) Find the zeros of $T_n(x)$ on $[-1, 1]$.

**Soln:** We know that the zeros of a the cosine function occur when $n \theta = \frac{(2j+1)\pi}{2}$ for
$j = 0, \ldots, n$. Thus the roots of $T_n$ occur at $x = \cos \left( \frac{(2j+1)\pi}{2n} \right)$ for $j = 0, \ldots, n$.

iii. (15 points) Show that $T_n$ are orthogonal with respect to the weight $1/\sqrt{1-x^2}$, and
compute their weighted $L^2$ norms for $n = 0, 1, 2, \ldots$

**Soln:** Define the change of variables $x = \cos(\theta)$ for $\theta \in [0, \pi]$. Then, note that
d$x = -\sin(\theta)d\theta$, and that $\sqrt{1-x^2} = \sqrt{1-\cos^2(\theta)} = \sin(\theta)$. Then,
$$\int_{-1}^{1} T_i(x)T_j(x) \frac{dx}{\sqrt{1-x^2}} = -\int_{0}^{\pi} T_i(x)T_j(x)d\theta.$$  
To show orthogonality using the latter integral, use the definition of Chebyshev polynomials
$$\int_{0}^{\pi} T_i(x)T_j(x)d\theta = \int_{0}^{\pi} \cos(i\theta) \cos(j\theta)$$
and the cosine product formula
$$\int_{0}^{\pi} \cos(i\theta) \cos(j\theta) = \frac{1}{2} \int_{0}^{\pi} \cos((i-j)\theta) + \cos((i+j)\theta)d\theta.$$  
For $i \neq j$, the above expression gives $\frac{1}{2} (\sin((i-j)\pi) + \sin((i+j)\pi)) = 0$. For
$i = j > 0$, the integral instead evaluates to
$$\frac{1}{2} \int_{0}^{\pi} 1 + \cos(2i\theta)d\theta = \frac{\pi}{2}.$$  
In the case that $i = j = 0$, $(T_0, T_0) = \pi$. Thus $\|T_i\|^2 = \pi/2$ if $i > 0$ and $\|T_0\|^2 = \pi$. 
(3) (15 points) Let $[a, b]$ be any fixed interval. Given any $\epsilon > 0$, show that there exists some $f \in C([a, b])$ such that

$$\|f\|_{L^2} \leq \epsilon \quad \text{while} \quad \|f\|_{L^\infty} \geq 1/\epsilon.$$  

[Süli and Mayers, problem 8.1]

**Soln:** Let $g(x) = (f(x))^2$. Now our goal is to find $g(x) \geq 0$ such that $\int_a^b g(x)dx$ and $\max_{x \in [a, b]} |g(x)| \geq 1/\epsilon^2$.

We choose to define $g(x)$ as a piecewise continuous function given by

$$g(x) = \begin{cases} \frac{1}{\epsilon^2(b-a-\epsilon^3)}(x - a - \epsilon^3) & \text{for } a < x < a + \epsilon^3 \\ 0 & \text{for } a + \epsilon^3 \leq x \leq b \end{cases}$$

Figure 0.2 provides a plot of $g(x)$. Noticing that the integral is simply the area of a triangle is how this function is constructed.

Then

$$f(x) = \begin{cases} \sqrt{\frac{1}{\epsilon^2(b-a-\epsilon^3)}(x - a - \epsilon^3)} & \text{for } a \leq x < a + \epsilon^3 \\ 0 & \text{for } a + \epsilon^3 \leq x \leq b \end{cases}$$

Figure 0.2. Illustration of the function $g(x)$ in problem 2.
(4) Consider approximations to $x$ for $x \in [0, 1]$.

i. (10 points) Find the line that best approximates $\sqrt{x}$ in the minimax ($L^\infty$) sense, and report the error. Hint: Consider the oscillation theorem, and the derivative of the error at its extreme points.

**Soln:** Let $p_1(x) = \alpha + \beta x$ and $\delta$ denote the maximum error. We know the absolute maximum error occurs at $x_0 = 0$, $x_2 = 1$ and some unknown $x_1 \in (x_0, x_2)$. We also know by the oscillation theorem that the error at $x_1$ will be opposite sign of that at $x_0$ and $x_2$. In equation form this means

$$
\delta = f(0) - p_1(0) = -\alpha,
\delta = f(1) - p_1(1) = 1 - \alpha - \beta,
-\delta = f(x_1) - p_1(x_1) = \sqrt{x_1} - \alpha - \beta x_1.
$$

There is one more fact we have to exploit. Since $x_1$ is a either a maximum or minimum of the the error function $E(x) = f(x) - p_1(x)$, we know $E'(x_1) = 1 - x_1^{-1/2} - \beta = 0$.

Solving for the unknowns, we find $\alpha = \frac{1}{8}$, $\beta = 1$, $\delta = -\frac{1}{8}$, and $x_1 = \frac{1}{4}$. Thus the $L^\infty$-approximation is $p_1(x) = \frac{1}{8} + x$.

ii. (10 points) Find the line that best approximates $\sqrt{x}$ in the least-square ($L^2$) sense, and report the error.

**Soln:** Let $\phi_0 = 1$ and $\phi_1 = x$. From class, we know the $L^2$-approximation is $p_1(x) = c_0 \phi_0(x) + c_1 \phi_1(x)$ where $c_0$ and $c_1$ are solutions to the linear system

$$
\begin{bmatrix}
< \phi_0, \phi_0 > & < \phi_0, \phi_1 > \\
< \phi_1, \phi_0 > & < \phi_1, \phi_1 >
\end{bmatrix}
\begin{bmatrix}
 c_0 \\
 c_1
\end{bmatrix}
=
\begin{bmatrix}
< f, \phi_0 > \\
< f, \phi_1 >
\end{bmatrix}.
$$

Now

$$
< \phi_0, \phi_0 > = \int_0^1 1 \, dx = 1
$$
$$
< \phi_0, \phi_1 > = < \phi_1, \phi_0 > = \int_0^1 x \, dx = 1/2
$$
$$
< \phi_1, \phi_1 > = \int_0^1 x^2 \, dx = 1/3
$$

and

$$
< \phi_0, f > = \int_0^1 \sqrt{x} \, dx = 2/3
$$
$$
< \phi_1, f > = \int_0^1 x^{3/2} \, dx = 2/5
$$

Solving for $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ in (0.1), we find $p_1(x) = \frac{4}{15} + \frac{4}{5} x$ and the $L^2$-error in our approximation is

$$
\| f - p_1 \|_{L^2}^2 = \int_0^1 \left( \sqrt{x} - \left( \frac{4}{15} + \frac{4}{5} x \right) \right)^2 \, dx
= \frac{131}{1350} = 0.097037.
$$
iii. (10 points) For a general interval $[a, b]$, prove that for all $f \in C([a, b])$,

$$
\min_{p \in P_n} \|f - p\|_{L_2} \leq \sqrt{b - a} \min_{p \in P_n} \|f - p\|_{L_\infty}.
$$

Confirm that your solutions to part i. and part ii. are consistent with this bound.

**Soln:**

$$
\min_{p \in P_n} \|f - p\|_{L_2}^2 = \min_{p \in P_n} \int_a^b (f(x) - p(x))^2 \, dx
\leq \min_{p \in P_n} \int_a^b \left( \max_{x \in [a, b]} |f(x) - p(x)| \right)^2 \, dx
= \min_{p \in P_n} (\|f - p\|_{L_\infty})^2 \int_a^b \, dx
= \min_{p \in P_n} (\|f - p\|_{L_\infty})^2 (b - a)
$$

Taking the square root of both sides yields the desired result.