$$\left.\begin{aligned}(n+1) &\Leftarrow p_{2n+1}(x_i) = f(x_i)\\ (n+1) &\Leftarrow p'_{2n+1}(x_i) = f'(x_i)\end{aligned}\right\} \quad i=0,\dots,n$$

$$\begin{array}{c} \\ n+1 \\ n+1 \\ \end{array}\underbrace{\begin{bmatrix} V \\ \hline \partial V \end{bmatrix}}_{2n+2}\begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_n) \\ \hline f'(x_0) \\ \vdots \\ f'(x_n) \end{bmatrix}$$

Error est.: For error at point t

Thm: $f \in C^{2n+2}(I_t)$ w/ I_t = smallest interval containing both t & $\{x_i\}_{i=0}^{n}$.

$\Rightarrow$ Then $\exists\, \alpha \in I_t$ s.t.

$$e(t) = f(t) - p_{2n+1}(t) = \frac{f^{(2n+2)}(\alpha)}{(2n+2)!}\psi(t)^2$$

w/ $\psi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

Pf: Define

$$G(x) = e(x) - \left(\frac{\psi(x)}{\psi(t)}\right)^2 e(t)$$

w/ $\psi(x) = \prod_{i=0}^{n}(x-x_i)$

$$G(x_i) = \underbrace{\cancel{e(x_i)}}_{\substack{f(x_i) - p_{2n+1}(x_i) \\ = 0}} - \frac{\cancel{\psi(x_i)}}{\psi(t)} e(t) = 0$$

for $i = 0, \dots, n$

$G(t) = 0 \Rightarrow n+2$ total roots

<u>MVT/Rolle's</u>: $G'(x)$ has $n+1$ roots in (x_i, x_{i+1}) (or t)

But we know that $G(x)$ has $2n+2$ total roots (counting double)

$\Rightarrow (n+1)$ roots of $G(x)$, $(n+1)$ roots of $G'(x)$ both at $x_0, \dots, x_n$.

$\Rightarrow G'(x_i)$ has $2n+2$ roots & roots are distinct.

$G''(x_i)$ has $2n+1$ distinct roots

$G'''(x_i)$ " $2n$ " "

$\vdots$

$G^{(2n+2)}(x)$ has only 1 root

$\hookrightarrow$ call it $\alpha \in I_t$

$$G^{(2n+2)}(x) = 0 = \frac{\partial^{2n+2}}{\partial x^{2n+2}}\left(e(t) - \left(\frac{\psi(x)}{\psi(t)}\right)^2 e(t)\right)$$

$$G^{(2n+2)}(\alpha) = f^{(2n+2)}(\alpha) - (\dots)$$

$$\psi(x)^2 = (x^{n+1} - \dots)^2$$
$$= (x^{2n+2} - \dots)$$

$$\Rightarrow \frac{\partial^{2n+2}}{\partial x^{2n+2}} \psi^2(x) = (2n+2)!$$

$$\Rightarrow G^{(2n+2)}(\alpha) = f^{(2n+2)}(\alpha) - \frac{(2n+2)!\, e(t)}{\psi(t)^2}$$

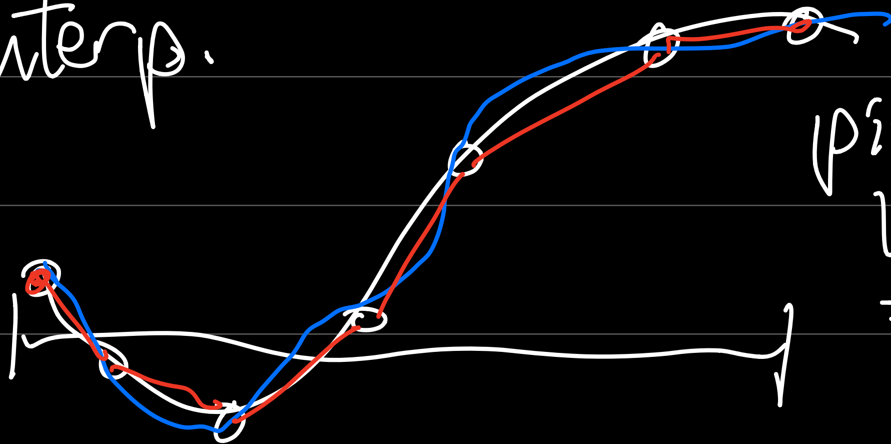
$$e(t) = \frac{f^{(2n+2)}(\alpha)}{(2n+2)!} \psi(t)^2$$

Note $\{x_i\}_{i=0}^{n}$ only impact $\psi(x)$!

- Polynomial interp. }
- Hermite interp. } both still have Runge's phenomena

= unavoidable using equally spaced points.

Alternative: piecewise polynomial interp.



piecewise linear = connect dots.

$= S_0(x)$

Previously: $\max\limits_{x \in [x_0, x_n]} |f(x) - S_0(x)|$

Note $e(t) = \dfrac{f''(\alpha)}{2}(t-x_0)(t-x_1)$ over each interval (x_i, x_{i+1})

$\alpha \in (x_i, x_{i+1}) \leftarrow$

$$\max_{(x_i,x_{i+1})}|e(x)| \le \max_{(x_i,x_{i+1})}\frac{|f''(x)|}{2}\max_{(x_i,x_{i+1})}|(x-x_0)(x-x_1)|$$

<u>Thm</u>:
$$\max_{[x_0,x_n]}|e(x)| \le \max_{[x_0,x_n]}\left|\frac{f''(x)}{8}\right| h^2$$

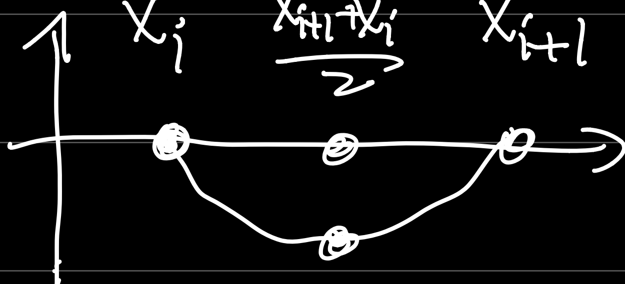
$$h = \max_{i\in(0,n-1)}|x_{i+1}-x_i|$$

<u>Pf</u>: on $[x_i, x_{i+1}]$

$$|f(x)-S_0(x)| = \left|\frac{f''(\alpha_i)}{2}(x-x_i)(x-x_{i+1})\right|$$

$$\le \left|\frac{f''(\alpha_i)}{2}\right| \underbrace{|(x-x_i)(x-x_{i+1})|}$$

b/c
$x \in [x_i, x_{i+1}]$



$$\left(\frac{x_{i+1}-x_i}{2}\right)\left(\frac{x_i-x_{i+1}}{2}\right)$$

$$= -\frac{h_i^2}{4}$$

$h_i = x_{i+1} - x_i$

$$|f(x) - S_0(x)| \le \max_{[x_i, x_{i+1}]} \frac{|f''(x)|}{2} \frac{h_i^2}{4}$$

$$\le \max_{[x_i, x_{i+1}]} \frac{|f''(x)|}{8} h_i^2$$

$$\max_{[x_0, x_n]} |f(x) - S_0(x)| \le \max_{[x_0, x_n]} \frac{f''(x)}{8} h^2$$

Why Chebyshev points are so good?

$\rightarrow$ minimax theory

minimize $\max_{[x_0, x_n]} |\psi(x)|$

grows fast $\leftarrow$ $\dfrac{f^{(n+1)}(\alpha)}{(n+1)!} \psi(t) \rightarrow 0$ faster