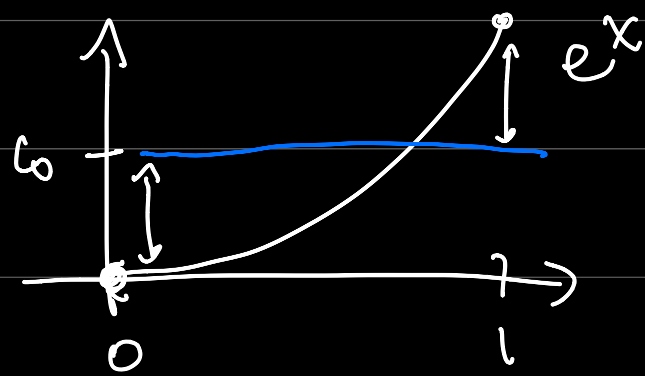
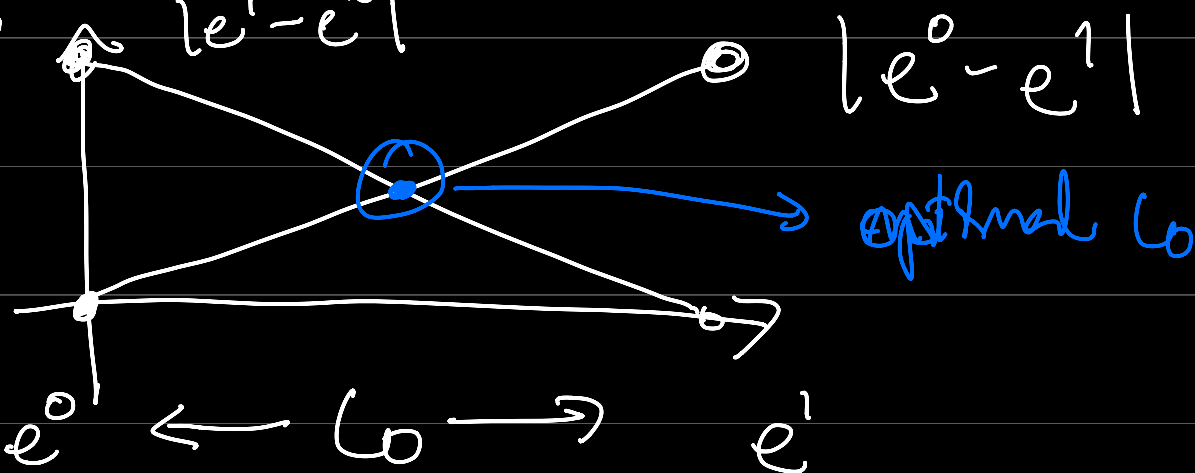
$$\min_{p \in P^n} \max_{x \in [a,b]} |f(x) - p(x)| = \min_{p \in P^n} \|f - p\|_{L^\infty}$$

Simple example:



e^x monotonically increasing, so

$$\max_{x \in [0,1]} |c_0 - e^x| = \max\{|c_0 - e^0|, |c_0 - e^1|\}$$



$$c_0 = \frac{e^0 + e^1}{2}$$

$\Rightarrow$ $e^x - \left(\frac{e^0 + e^1}{2}\right)$ error at $x = 0, 1$ is the same in magnitude

$x=0$: $e^0 - \left(\frac{e^0+e^1}{2}\right) = \frac{e^1-e^0}{2}$

$x=1$: $e^1 - \left(\frac{e^0+e^1}{2}\right) = \frac{e^0-e^1}{2}$

errors flip sign but are the same magnitude

$\Rightarrow$ characterize minimax-type approximations.

<u>Thm</u>: de la Vallée Poussin

If $f \in C[a,b]$, $r \in P^n$

& there $\exists$ $n+2$ points on $[a,b]$

$a \leq x_0 < x_1 < x_2 < \dots x_{n+1} \leq b$

where $f(x) - r(x)$ oscillates

sign $\Rightarrow$ $\text{sign}(f(x_j) - r(x_j))$

$= -\text{sign}(f(x_{j+1}) - r(x_{j+1}))$

for $j = 0, \dots, n$

$$\Rightarrow \min_{p \in P^n} \| f - p \|_{L^\infty} \geq \min_j | f(x_j) - r(x_j) |$$

Pf: Let p^* = (unique) minimax polynomial

$$\| f - p^* \|_{L^\infty} \geq \min_{x_j} | f(x_j) - r(x_j) |$$

Assume opposite:

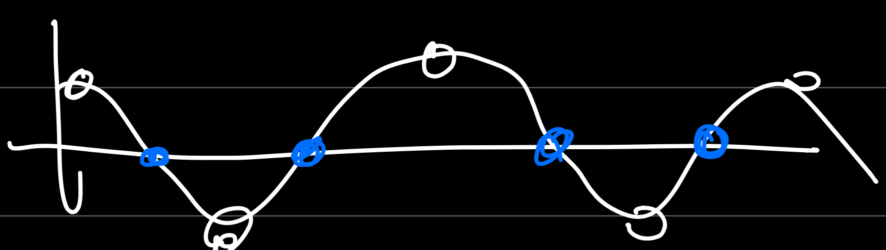
(✱) $\| f - p^* \|_{L^\infty} < | f(x_j) - r(x_j) | \ \forall x_j$

Consider diff b/w p^* & r at x_j

$$p^* - r = \underbrace{(f - r)}_{\text{by } ✱ \text{ = larger magnitude}} - (f - p^*)$$

$$\underbrace{\operatorname{sign}(p^*(x_j) - r(x_j))}_{\text{diff. b/w 2 polynomials}} = \underbrace{\operatorname{sign}(f(x_j) - r(x_j))}_{\text{oscillates } n+2 \text{ times.}}$$

$p^*(x) - r(x)$ oscillates $n+2$ times



$\Rightarrow p^*(x) - r(x)$ has $n+1$ roots.

$\Rightarrow p^* - r = 0$. But this contradicts (★) $\Rightarrow \exists j$ s.t.

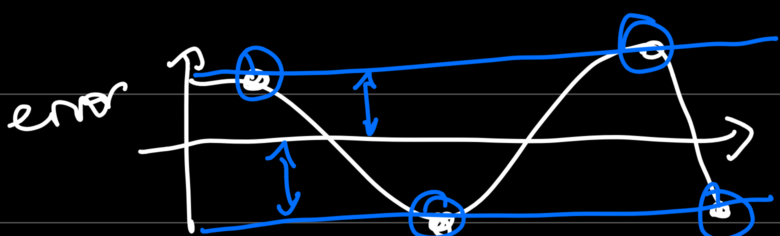
$$\|f - p^*\|_{L^\infty} \geq |f(x_j) - r(x_j)|$$

<u>Thm</u>: the Osc. Theorem (Suli/Mayers 8.3)

$\Rightarrow p^*$ is the minimax approx to $f(x) \in C[a,b]$

<u>iff</u> $|f(x_j) - p^*(x_j)| = \|f - p_*\|_{L^\infty}$

at $x_0, \dots, x_{n+1} \Rightarrow n+2$ distinct points.



$\Rightarrow$ exact conditions for minimax.

<u>Thm</u>: p^* is unique (pf. relies on showing $n+2$ equioscillation pts are also unique)

<u>Ex</u>: $f(x) = e^x \Rightarrow$ compute $p^* \in P^1$

seeks $x_0, x_1, x_2 \in [0,1]$ at which

$$\|f - p^*\| = |\delta|$$

$\to f(x_0) - p^*(x_0) = \delta$

$f(x_1) - p^*(x_1) = -\delta$

$\longrightarrow f(x_2) - p^*(x_2) = \delta$

$p^*(x) = \alpha + \beta x$. By e^x is monotonic increasing $\Rightarrow$ extrema at $0, 1$

$x_0 = 0$, $x_2 = 1$ & δ = error at x_0, x_2

$f(x_0) - p^*(x_0) = \cancel{e}^{1} - \alpha = \delta$ ① $(x = x_0 = 0)$

$f(x_2) - p^*(x_2) = e^1 - \alpha - \beta = \delta$ ② $(x = x_2 = 1)$

3 unknowns $\Rightarrow \alpha, \beta, \delta$

$f(x_1) - p^*(x_1) = e^{x_1} - \alpha - \beta x_1 = -\delta$ ③

By ① + ② $\Rightarrow 1 - \alpha = \delta = e^1 - \alpha - \beta = \delta$ $\qquad 1 - \cancel{\alpha} = e^1 - \cancel{\alpha} - \beta$

$\boxed{\beta = e^1 - 1}$

By ① + ③ $\Rightarrow 2\alpha = e^{x_1} - x_1(e^1 - 1) + 1$

How to determine x_1 ??

Use that at x_1, error = max.

& note that for e^x, the error differentiable

$|f(x_1) - p^*(x_1)| = \max\limits_{x \in [a,b]} |f - p^*| \Rightarrow$ find max by diff.

$$\frac{\partial}{\partial x}\left(e^{x_1} - (\alpha + \beta x_1)\right) = e^{x_1} - \beta = 0$$

$$\beta = e^{x_1} \Rightarrow x_1 = \log(\beta)$$

$$\boxed{x_1 = \log(e^1 - 1)}$$

from before

$$2\alpha = e^{x_1} - x_1(e^1 - 1) + 1$$

$$\alpha = \frac{1}{2}\left(e^{x_1} - x(e^1 - 1) + 1\right) \approx .89406$$

$$\beta = (e^1 - 1) \approx 1.71828$$

Aux quantities

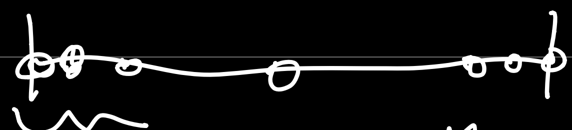
$$\delta = .10593$$

$$x_1 = \log(e^1 - 1) \approx .541$$

$\Rightarrow$ Remez algorithm (iterative way to find n+2 equioscillating pts)

Applying minimax: good interp points.

$x_0, \dots, x_n$ = equispaced

$x_0, \dots, x_n$ 

clusters = better errors

Want to find $x_0, \dots, x_n$ s.t.

$\|f - p\|_{L^\infty}$ is small

$$|f(x) - p(x)| \le \left|\frac{f^{(n+1)}(\alpha)}{(n+1)!}\right| \left|\prod_{j=0}^{n}(x - x_j)\right|$$

$$\max|f - p| \le \max_{x\in[a,b]}\left(|\;|\right) \underbrace{\max_{x\in[a,b]}\left|\prod_{j=0}^{n}(x - x_j)\right|}$$

choose $x_0, \dots, x_n$ minimize this!