Recall min-max approximation theorem:

<u>Thm</u>: Let $f \in C[a,b]$, $p \in P^n$, and assume

$$\max_{x\in[a,b]} |f(x)-p(x)| = \|f-p\|_{L^\infty} = \delta$$

If $f(x)-p(x)$ equioscillates between $\pm\delta$ $n+2$ times, $p(x)$ is the minimax approximation to $f(x)$.

$\Rightarrow$ Use to find good interpolation points.

Recall for $f \in C^{n+1}([a,b])$, if $p \in P^n$ is the degree n interpolant, then for $x, \alpha \in [x_0, x_n]$

$$f(x) - p(x) = \frac{f^{(n+1)}(\alpha)}{(n+1)!} \prod_{j=0}^{n} (x-x_j)$$

$$\Rightarrow \max_x |f(x)-p(x)| \leq \max_x \left|\frac{f^{(n+1)}(x)}{(n+1)!}\right| \underbrace{\max_x \left|\prod_{j=0}^{n}(x-x_j)\right|}$$

find x_j to minimize this!

$\Rightarrow$ If x_j are the roots of the degree $n+2$ Chebyshev polynomial, they are optimal.

$\Rightarrow (x-x_0)(x-x_1)\cdots(x-x_n) = x^{n+1} - r(x)$

$r(x) \in P^n$

$\Rightarrow$ Find $\underbrace{r(x)}_{\text{depends indirectly on } x_j}$ which minimizes $x^{n+1} - r(x)$

Let $[-1,1]$. Define nth chebyshev polynomial

<u>Def:</u> $T_n(x) = \cos(n \cos^{-1}(x))$

$$T_0(x) = \cos(0 \cos^{-1}(x)) = \cos(0) = 1$$

$$T_1(x) = \cos(\cos^{-1}(x)) = x$$

$$T_2(x) = \cos(2\cos^{-1}(x))$$

$\cos(2\theta) = 2\cos(\theta)^2 - 1$

$$= 2\cos(\cos^{-1}(x))^2 - 1$$

$$= 2x^2 - 1$$

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$

<u>Properties</u>: for $x \in [-1,1]$

1. $|T_n(x)| \le 1$
2. Roots are at $x_j = \cos\left(\frac{(2j-1)\pi}{2n}\right)$ $\overbrace{j=1,\dots,n}^{n \text{ roots}}$ of $T_n(x)$
3. $|T_n(x)|$ is maximized at $\eta_j = \cos\left(\frac{j\pi}{n}\right)$ $j = 0,\dots,n$

$$T_n(\eta_j) = (-1)^j$$ $n+1$ points.

Recall: $x^{n+1} - r(x)$

but $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$

$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 + \cdots$$
$$T_4(x) = 8x^4 + \cdots$$
$$\vdots$$
$$T_{n+1}(x) = 2^n x^{n+1} + \cdots$$

Normalize by $2^{-n} \Rightarrow \hat{T}_{n+1}(x) = 2^{-n} T_{n+1}(x)$

Implies $\hat{T}_{n+1}(x) = x^{n+1} - r(x)$

Note that $\underbrace{\hat{T}_{n+1}(\eta_j) = 2^{-n}(-1)^j}$ for $j = 1, \dots, n+2$

$|\eta_j^{n+1} - r(\eta_j)| = 2^{-n}(-1)^j$ oscillates $n+2$ times

$r(x) = x^{n+1} - \hat{T}_{n+1}(x)$ = minimax approx to x^{n+1} !!

Need $n+1$ diff pts.

$\hookrightarrow$ need roots of a degree $n+1$ polynom.

Find roots of $\hat{T}_{n+1}(x) = x^{n+1} - r(x) \Rightarrow x_j = \cos\left(\frac{(2j-1)\pi}{2n}\right)$

$j = 1, \dots, n+1$

Recap: find x_j to minimize $\prod_{j=0}^{n}(x-x_j) \iff$ finding $r(x)$ to min. $x^{n+1}-r(x)$

$\hat{T}_{n+1}(x) = x^{n+1}-r(x) \Rightarrow$ $n+2$ maxima of $\hat{T}_{n+1}$ are equioscillation pts.

$\Rightarrow \prod_{j=0}^{n}(x-x_j) = \hat{T}_{n+1}(x)$ when x_j minimize $\left|\prod_{j=0}^{n}(x-x_j)\right|$

roots of $\hat{T}_{n+1}(x)$ determine x_j up to scaling.