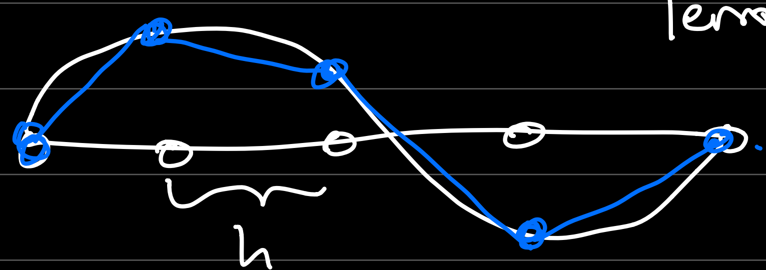
Overview of material so far:

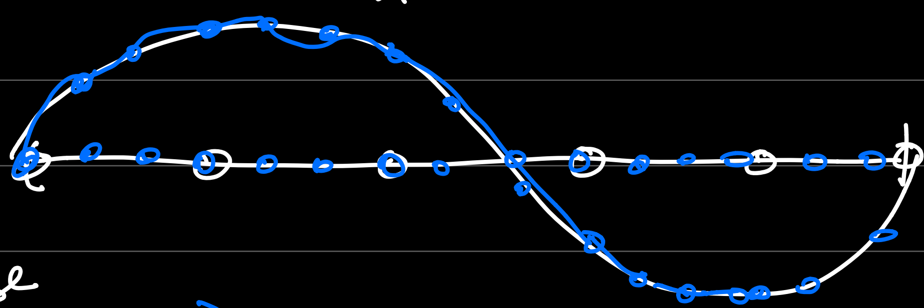
- polynomial interpolation (Lagrange & Hermite)
- Lagrange & Newton bases
- minimax approximation theory & optimal interpolation points.
- piecewise polynomial interpolation.
  - piecewise linear in class
  - piecewise cubic on homework

Piecewise Lagrange interp.

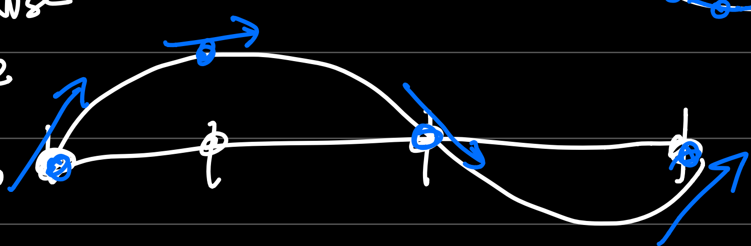


$|\text{error}| \leq C h^2$

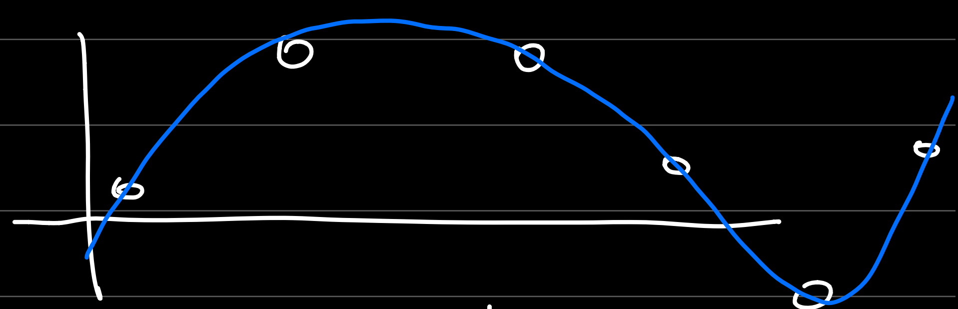
$C \to \dfrac{\max|f''(x)|}{8}$



Piecewise Hermite interp.



Spline interpolation



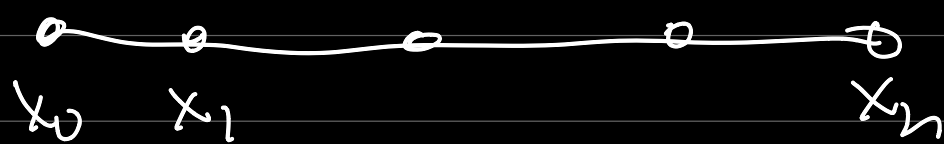
$\to$ interpolate function values but get a smoothly differentiable function.

Spline: global piecewise polynomial approximations with some specified level of continuity.

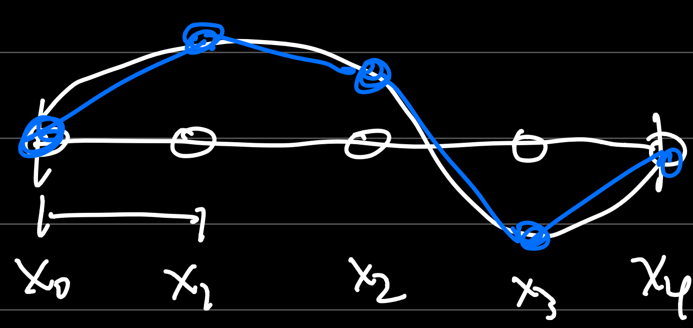
<u>Def</u>: a spline $S(x)$ of order k & degree p satisfies

① $S(x)$ is a degree n polynomial on $[x_i, x_{i+1}]$

② $\frac{\partial^r S}{\partial x^r}$ is continuous on $[x_0, x_n]$ $0 \le r \le k$



<u>Note</u>: If $p = 1$



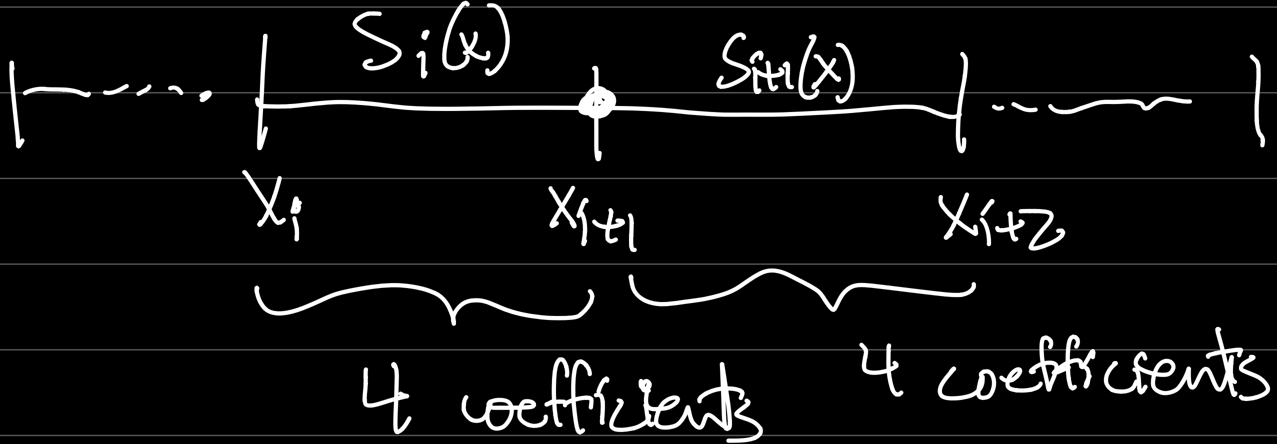
$\Rightarrow$ spline of degree $p=1$ & order $k=0$.

$\Rightarrow$ $k \le p-1$. For this lecture, assume $k = p-1$ (maximal continuity splines)

Ex: $p = 3$ (Cubic spline) $\Rightarrow k = 2$

define

$$S_i(x) = S(x)\Big|_{[x_i, x_{i+1}]} \in P^3 \quad \text{on } [x_i, x_{i+1}]$$



$k = 2 \Rightarrow S, S', S''$ are all continuous.

$$\Rightarrow \quad \begin{aligned} S_i(x_{i+1}) &= S_{i+1}(x_{i+1}) \\ S_i'(x_{i+1}) &= S_{i+1}'(x_{i+1}) \\ S_i''(x_{i+1}) &= S_{i+1}''(x_{i+1}) \end{aligned}$$

$n+1$ points $\Rightarrow$ n intervals.

over each interval = 4 coeffs

$= 4n$ coeffs

at each interior point x_i $(n-1)$

$\Rightarrow$ $3n-3$ constraints.

$4n - (3n-3) = n+3$ free parameters.

$\Rightarrow$ but we have $n+1$ total points

$\Rightarrow$ 2 free parameters / coefficients which cannot be determined just by interpolation!!

Need two additional constraints!

Common conditions

① Natural spline: $S''(x_0) = S''(x_n) = 0$

② "Complete" spline: $S'(x_0)$, $S'(x_n)$ specified.

③ Modify continuity conditions near endpoints (closed splines)

minimize curvature over all possible C^2 functions (Numerische Mathematik)

$\Rightarrow$ In general, splines need extra conditions in order to interpolate at $n+1$ points.

$\Rightarrow$ How many conditions? Can counting the number of basis functions for a spline.

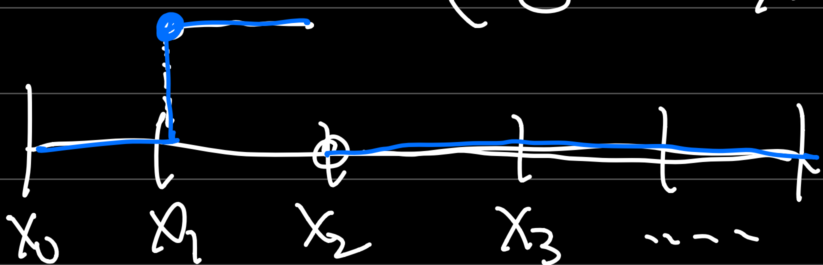
$\rightarrow$ B-splines: convenient basis representation for spline $S(x)$

$B_{i,p}(x)$ $\quad i =$ index, $p =$ degree $(k = p-1)$

Recursive def'n

$$B_{i,0}(x) = \begin{cases} 1 & x \in [x_i, x_{i+1}) \\ 0 & \text{o/w} \end{cases}$$

$B_{1,0}(x)$



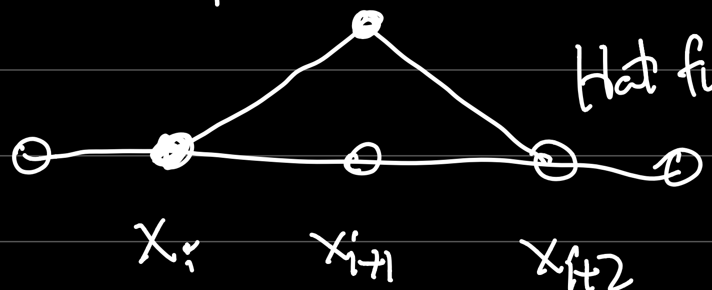
$$B_{i,p}(x) = \frac{(x - x_i)}{(x_{i+p} - x_i)} B_{i,p-1}(x) + \frac{(x_{i+p+1} - x)}{(x_{i+p+1} - x_{i+1})} B_{i+1,p-1}(x)$$

Properties:

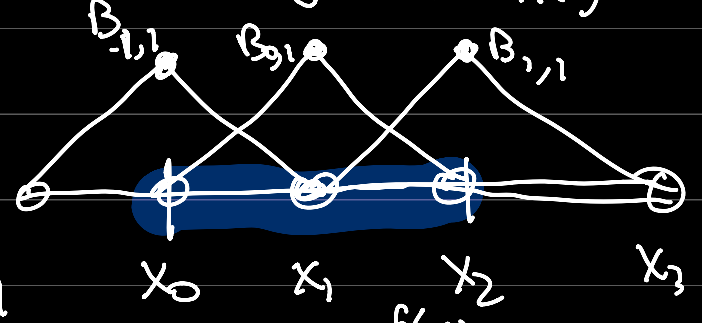
1. $B_{i,}(x) \in C^{p-1}(\mathbb{R})$
2. $B_{i,p}(x) = 0$ if $x \notin (x_i, x_{i+n+1})$
3. $B_{i,p}(x)$ is polynomial over $[x_i, x_{i+1})$

Ex: $p = 1 \Rightarrow B_{i,1}(x) = \begin{cases} \dfrac{x - x_i}{x_{i+1} - x_i} & \text{on } [x_i, x_{i+1}) \\ \dfrac{(x_{i+n+1} - x)}{(x_{i+n+1} - x_{i+1})} & \text{on } [x_{i+1}, x_{i+2}) \end{cases}$

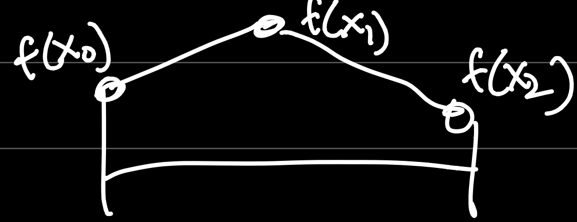
Hat function

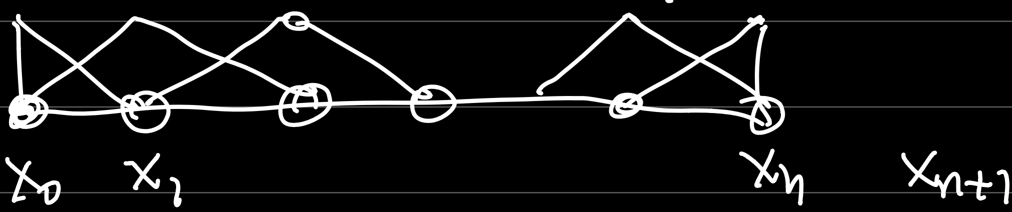


$B_{i,1}(x)$



$\sum_{i=1}^{3} B_{i,1}(x)\, f(x_{i+1}) =$





n+1 basis fxns from n+3 pts.