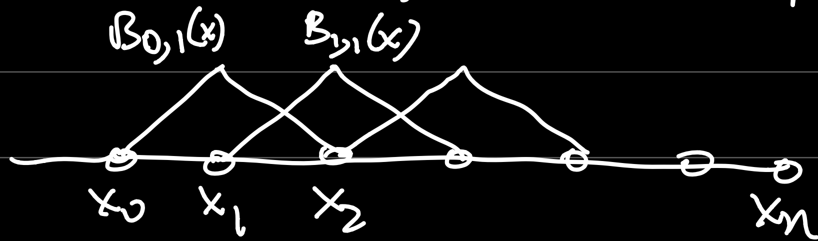
B-splines : piecewise polynom. which are globally in $C^k([a,b])$ where k = order of the spline.

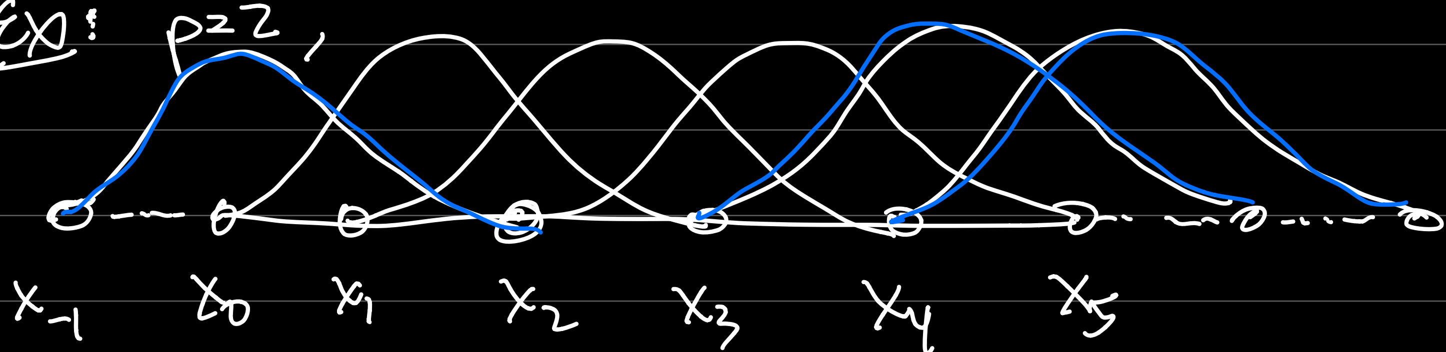
Given $x_0, \ldots, x_n$, $B_{i,0} = \begin{cases} 1 & x \in [x_i, x_{i+1}) \\ 0 & \text{o/w} \end{cases}$

$$B_{i,p}(x) = \frac{(x - x_i)}{(x_{i+p} - x_i)} B_{i,p-1}(x) + \frac{(x_{i+p+1} - x)}{(x_{i+p+1} - x)} B_{i+1,k-1}(x)$$

Ex : $p = 1$



Ex : $p = 2$,



Mismatch b/w # points & # splines which are easily defined on an interval.

$\Rightarrow$ need extra conditions to interpolate w/ splines.

Look @ matrices for spline interp. at "knots."

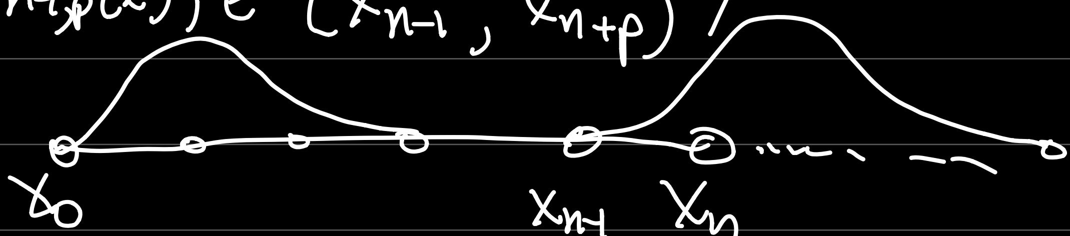
$(x_0, \ldots, x_n)$ referred to as "knots"

$$S(x_j) = \sum_{i=-\infty}^{\infty} c_i B_{i,p}(x) = f(x_j) \quad j = 0, \ldots, n$$

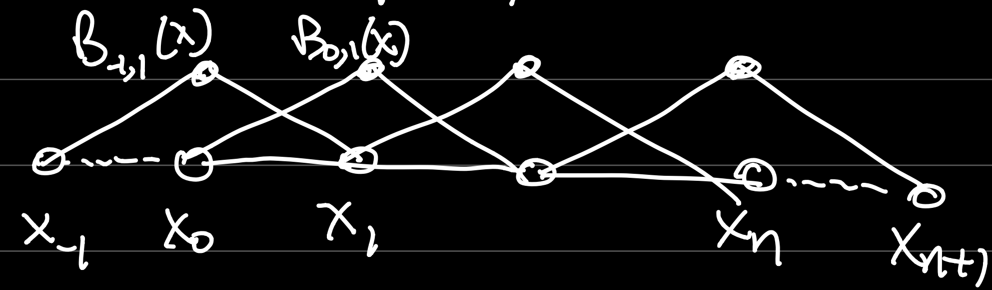
$$\operatorname{supp}(B_{i,p}(x)) \in (x_i, x_{i+p+1})$$

$$S(x_j) = \sum_{i=-p}^{n-1} c_i B_{i,p}(x) \qquad \text{for } k>0.$$

$$\left.\begin{array}{l} \operatorname{supp}(B_{-p,p}(x)) \in (x_0, \dots, x_{p+1}) \\ \operatorname{supp}(B_{n-1,p}(x)) \in (x_{n-1}, x_{n+p}) \end{array}\right\} \text{ splines which lie within } [x_0, x_n].$$



Ex: $p = 1$



$$\sum B_{i,1}(x_j)\, c_i = f(x_j)$$

$$\begin{bmatrix} B_{-1,1}(x_0) & B_{0,1}(x_0) & \cdots & B_{n-1,1}(x_0) \\ B_{-1,1}(x_1) & B_{0,1}(x_1) & \cdots & B_{n-1,1}(x_1) \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$\Rightarrow \quad c_i = f(x_{i+1}) \qquad i = -1, \dots, n-1$$

Ex : $k=2$. Assume $x_0, \dots, x_n$ equally spaced and $x_{i+1} - x_i = h$

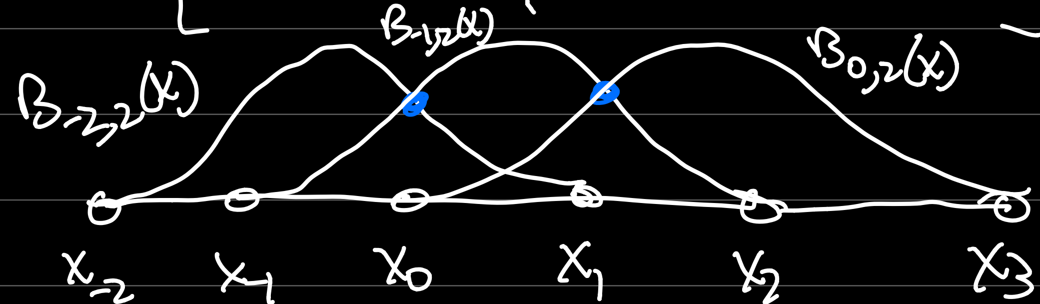
$$B_{i,2}(x) = \frac{(x-x_i)}{(x_{i+2}-x_i)} B_{i,1}(x) + \frac{(x_{i+3}-x)}{(x_{i+3}-x_{i+1})} B_{i+1,1}(x)$$

x_{i+1}, x_{i+2}

$$= \frac{(x-x_i)}{2h} B_{i,1}(x) + \frac{(x_{i+3}-x)}{2h} B_{i+1,1}(x)$$

$$\boxed{\tfrac{1}{2} = B_{i,2}(x_{i+1})} = \tfrac{1}{2} B_{i,1}(x_{i+1})^{1} + (\cdots)\, \cancel{B_{i+1,1}(x_{i+1})}^{0}$$

$$\begin{bmatrix} B_{-2,2}(x_0) & B_{-1,2}(x_0) & \cdots & B_{n-1,2}(x_0) \\ B_{-2,2}(x_1) & B_{-1,2}(x_1) & & \vdots \\ \vdots & B_{-1,2}(x_2) & & \vdots \\ & \vdots & & \end{bmatrix} = \text{interp. matrix.}$$



$$\begin{bmatrix} \alpha & \alpha & & & \\ & \alpha & \alpha & & \\ & & \alpha & \alpha & \\ & & & \alpha & \ddots \\ & & & & \alpha \end{bmatrix} \qquad \alpha = B_{-2,2}(x_0)$$

$$\Rightarrow \begin{bmatrix} \frac12 & \frac12 & & \\ & \frac12 & \frac12 & \\ & & \ddots & \frac12 \\ & & & \frac12 \end{bmatrix} \qquad p = 2$$

$$p=3 \quad \begin{bmatrix} x & x & x & & & & \\ & x & x & x & & & \\ & & x & x & x & & \\ & & & x & x & x & \\ & & & & x & x & x \end{bmatrix}$$

<u>Interpolation</u> : $\displaystyle p(x) = \sum_{i=0}^{n} \underbrace{\varphi_i(x)}_{\text{basis function}} c_i$

$p(x_j) = f(x_j) \quad (j=0,\dots,n)$

$$\underbrace{\begin{bmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \cdots \\ \varphi_0(x_1) & \varphi_1(x_1) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\text{square \& invertible}} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_n) \end{bmatrix}$$

$\Rightarrow$ Least squares approach, Suppose

$x_0, \dots, x_m$ & $n+1$ basis fxns $\{\varphi_i\}_{i=0}^{n}$

$m \geq n$

Suppose we want

$p(x_j) = f(x_j) \quad j=0,\dots,m$

$$\begin{array}{cc} m+1 & \begin{bmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \\ \varphi_0(x_m) & \varphi_1(x_m) & \cdots \end{bmatrix} \end{array} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

$n+1$ $\quad f(x_m)]$

$$A x = b, \quad A \in \mathbb{R}^{m \times n}, \quad m \geq n$$

Overdetermined $\Rightarrow$ cannot solve exactly.

Instead of solving overdetermined system exactly, try solving in a least squares sense.

$$\Rightarrow \quad \min_{x} \underbrace{\|Ax - b\|_2^2}$$

$$\Rightarrow \quad \min_{x} J(x)$$ objective fxn to min.

Solve $\dfrac{\partial}{\partial x_i} J(x) = 0 \iff \underbrace{A^T A x = A^T b}_{\text{normal equations.}}$

$$\frac{1}{m} A^T A x = \frac{1}{m} A^T b, \qquad A \in \mathbb{R}^{m \times n}$$

$$A = \begin{array}{c c} & j \\ i & \begin{bmatrix} & | & \\ - & \varphi_j(x_i) & - \\ & | & \end{bmatrix} \end{array} \qquad b = \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_m) \end{bmatrix}$$

$$\frac{1}{m+1}\left(A^T A\right)_{ij} = \frac{1}{m+1}\sum_{k=0}^{m} \varphi_i(x_k)\,\varphi_j(x_k) \approx \int_{x_0}^{x_m} \varphi_i(x)\varphi_j(x)$$

$$\frac{1}{m+1}\left(A^T b\right)_i = \frac{1}{m+1}\sum_{k=0}^{m} \varphi_i(x_k)\, f(x_k) \approx \int_{x_0}^{x_m} \varphi_i(x) f(x)$$

$\approx$