Last time: instead of solving a square matrix system for polynomial interpolation, what if we used $m > n+1$ points so that $p(x_j) = f(x_j)$ $j = 1, \dots, m$

Suppose $p(x) = \sum_{j=0}^{n} c_j \varphi_j(x)$ & $V_{ij} = \varphi_j(x_i)$

$\Rightarrow$ the interpolation system $\begin{bmatrix} V \end{bmatrix} \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} y \end{bmatrix}$ $(*)$

is over-determined

We can solve $(*)$ using least squares: $\min_c \|Vc - y\|_2^2$

$\Rightarrow$ c solves the normal equations $V^T V c = V^T y$

As $m \to \infty$, $\frac{1}{m} V^T V c = \frac{1}{m} V^T y$ looks like it converges to a system involving integrals.

$$\left.\begin{aligned} \frac{1}{m}(V^T V)_{ij} &\to \frac{1}{m}\sum_{k=1}^{m} \varphi_i(x_k)\varphi_j(x_k) \to \int_a^b \varphi_i(x)\varphi_j(x) \\ \frac{1}{m}(V^T y)_i &\to \quad \cdots \quad \to \int_a^b f(x)\varphi_i(x) \end{aligned}\right\}$$ theory for this system?

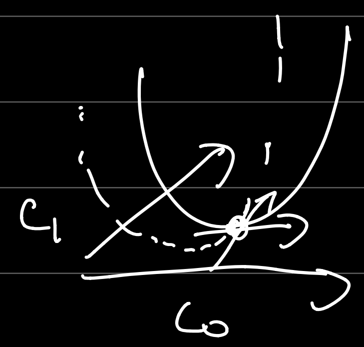
$\Rightarrow$ Minimizing polynomial with respect to the continuous L^2-norm

Ex: Given $f(x)$, find the linear polynomial $p(x)$ which minimizes

$$\frac{1}{2}\| p(x) - f(x) \|^2_{L^2([a,b])} = \frac{1}{2}\int_a^b (p-f)^2$$

$p_1(x) = c_0 + c_1 x$, so we just need to determine c_0 & c_1.

$$J(c_0, c_1) = \frac{1}{2}\int_a^b (c_0 + c_1 x - f(x))^2$$



Find c_0, c_1 s.t.

$$\frac{\partial J}{\partial c_0} = 0, \quad \frac{\partial J}{\partial c_1} = 0$$

① $\dfrac{\partial J}{\partial c_0} = \displaystyle\int_a^b (c_0 + c_1 x - f(x)) = 0$

② $\dfrac{\partial J}{\partial c_1} = \displaystyle\int_a^b (c_0 + c_1 x - f(x))\, x = 0$

① + ② $\quad c_0 \displaystyle\int_a^b 1 + c_1 \int_a^b x = \int_a^b f(x)$

$$c_0 \int_a^b x + c_1 \int_a^b x^2 = \int_a^b f(x)\, x$$

$$\Rightarrow \begin{bmatrix} \int_a^b 1 & \int_a^b x \\ \int_a^b x & \int_a^b x^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \int_a^b f(x) \\ \int_a^b f(x)\, x \end{bmatrix}$$

$$p(x) = \sum_{j=1}^{2} c_j \varphi_j(x) \qquad \begin{array}{l} \varphi_1(x) = 1 \\ \varphi_2(x) = x \end{array}$$

$$\Rightarrow \begin{bmatrix} \int_a^b \varphi_0(x)\varphi_0(x) & \int_a^b \varphi_0(x)\varphi_1(x) \\ \int_a^b \varphi_1(x)\varphi_0(x) & \int_a^b \varphi_1(x)\varphi_1(x) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \int_a^b f\varphi_0 \\ \int_a^b f\varphi_1 \end{bmatrix}$$

<u>Ex:</u> $$p(x) = \sum_{j=0}^{n} c_j \varphi_j(x)$$

$$\min_{c} \| f - p \|^2_{L^2([a,b])} = \min_{c_0,\dots,c_n} \underbrace{\| f - p \|^2_{L^2([a,b])}}_{J(c_0,\dots,c_n)}$$

find $c_0,\dots,c_n$ s.t. $\dfrac{\partial J}{\partial c_i} = 0$

$$\Rightarrow \frac{\partial}{\partial c_i} \int_a^b \left(f(x) - \sum_{j=0}^{n} c_j \varphi_j(x) \right)^2$$

$$\Rightarrow \int_a^b \left(f(x) - \sum_{j=0}^{n} c_j \varphi_j(x) \right) \varphi_i(x) = 0$$

for $i = 0,\dots,n$.

$$\Rightarrow G = \begin{bmatrix} \int_a^b \varphi_0\varphi_0 & \int_a^b \varphi_0\varphi_1 & \cdots & \int_a^b \varphi_0\varphi_n \\ \int_a^b \varphi_0\varphi_1 & \ddots & & \\ \vdots & & \ddots & \\ \int_a^b \varphi_0\varphi_n & & & \int \varphi_n\varphi_n \end{bmatrix} \rightarrow \text{ij}^{\text{th}} \text{ entry is } \int_a^b \varphi_i\varphi_j$$

$Gc = b$ w/ $b = \begin{bmatrix} \int_a^b f(x)\varphi_0(x) \\ \vdots \\ \int_a^b f(x)\varphi_n(x) \end{bmatrix}$, $b_i = \int_a^b f(x)\varphi_i(x)$

$\Rightarrow$ L^2 norm minimization is just one of a huge class of continuous least squares approximations

L^2 norm = one instance of an inner product norm.

Def: An inner product (IP) is a map from $V \times V \to \mathbb{R}$ s.t.

1. Bilinearity: Let $\alpha, \beta \in \mathbb{R}$. Then, if $f, g, h \in V$
$$(\alpha f + \beta g, h) = \alpha(f,h) + \beta(g,h)$$
2. Symmetry: $(f,g) = (g,f)$
3. Positivity: $(f,f) \geq 0$ & $(f,f) = 0$ if $f = 0$

Ex: L^2 inner product: $(f,g) = \int_a^b f(x)g(x)$
$$(f,f) = \int_a^b f^2 \geq 0$$
= inner product over (Lebesgue) integrable functions on $[a,b] \to L^2([a,b])$

Define $\|f\|_{L^2([a,b])} = \sqrt{(f,f)}$

Def: inner prod. norm $\|f\| = \sqrt{(f,f)}$

Recall: $\|\cdot\| : V \to \mathbb{R}$ is a norm if

① $\|f\| \geq 0$ & $\|f\| = 0$ iff $f = 0$

② $\alpha \in \mathbb{R}$, $\|\alpha f\| = |\alpha| \, \|f\|$

③ $\|f+g\| \leq \|f\| + \|g\|$

All IPs satisfy the Cauchy-Schwarz Inequality (CS)

$\Rightarrow (f,g) \leq \|f\| \, \|g\|$

Pf: Let $\alpha \in \mathbb{R}$. $\quad 0 \leq \|f - \alpha g\|^2 = (f - \alpha g, f - \alpha g)$

$= \|f\|^2 - 2\alpha (f,g) + \alpha^2 \|g\|^2$

Specify $\alpha = \dfrac{(f,g)}{\|g\|^2}$

$\Rightarrow \quad 0 \leq \|f\|^2 - \dfrac{2(f,g)^2}{\|g\|^2} + \dfrac{(f,g)^2}{\|g\|^2}$

$0 \leq \|f\|^2 - \dfrac{(f,g)^2}{\|g\|^2}$

$(f,g)^2 \leq (\|f\| \, \|g\|)^2 \Rightarrow (f,g) \leq \|f\| \|g\|$

Triangle ineq: use CS

$\Rightarrow \|f+g\|^2 = \|f\|^2 + 2(f,g) + \|g\|^2$

$\leq \|f\|^2 + 2\|f\| \, \|g\| + \|g\|^2$

$\Rightarrow \|f+g\| \leq \|f\| + \|g\|$

Ex: L^2 inner prod on $[a,b]$

Ex: weighted L^2 inner products on $[a,b]$

$$(f,g) = \int_a^b f(x)g(x)\,\omega(x) \quad \text{w/ } \omega(x) > 0$$

Ex: Sobolev inner product

$$(f,g)_{H^1([a,b])} = (f,g)_{L^2([a,b])} + (f',g')_{L^2([a,b])}$$

Differences b/w cont. lsq. & interpolation

- interp. still depends on point distribution
  cont. lsq is indep. of points, just depends on IP & basis functions $\varphi_j(x)$
- optimal in IP norm.
- cont. lsq. more involved.

Similarities: monomials = bad.

$$G_{ij} = \int_a^b \varphi_j(x)\varphi_i(x) \quad . \quad \text{If } \varphi_j(x) = x^j$$

$$\Rightarrow G_{ij} = \int_0^1 x^i x^j = \int_0^1 x^{i+j} = \left.\frac{x^{i+j+1}}{i+j+1}\right|_0^1$$

Hilbert matrix $= \dfrac{1}{i+j+1}$

$$\begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/5 & & & \\ \vdots & & 1/7 & & \\ \vdots & & & 1/9 & \\ 1/6 & & & & 1/11 \end{bmatrix} \end{array}$$