Continuous least squares

$$\min_{p} \|f - p\|^2 \quad (*)$$

→ focus of L^2 inner product

$$(f, g) = \int_a^b f(x) g(x)\, dx$$

→ if $p(x) = \sum_{j=0}^{n} c_j x^j$ & we want to solve (*)

$$\Leftrightarrow \quad \begin{array}{cc} & j \\ i & \underbrace{\begin{bmatrix} \quad \int_0^1 x^i x^j \quad \end{bmatrix}}_{G = \text{ Hilbert matrix}} \end{array} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \int_0^1 x^i f(x) \\ \\ \end{bmatrix} \leftarrow i$$

⇒ Monomials are a bad choice for contin. least squares! Find a basis where the contin. least squares problem is easy to solve (easy to invert G)

Polynom. interp: Lagrange polynomials

$$V^{-1} = I^{-1} = I$$

Contin. least squares: $G_{ij} = \int_a^b \varphi_i(x)\varphi_j(x) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

⇒ orthogonal polynomials (orthonormal)

(Hierarchical) orthog. polynomials. Find $\{\varphi_i\}_{i=0}^{n}$
s.t. $(\varphi_i, \varphi_j) = 0$ and φ_k = exactly of degree k.

$\Rightarrow$ depends on def. (f,g)
(ex: $\underbrace{L^2([a,b])}_{\text{usually assume } [-1,1]}$, L^2_w, Sobolev)

Ex: $(f,g) = \int_{-1}^{1} f(x) g(x)$ (L^2 IP)
Legendre polynomial on $[-1,1]$

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = x$$
$$\varphi_2(x) = 3x^2 - 1$$
$$(n+1)\varphi_{n+1}(x) = (2n+1)x\varphi_n(x) - n\varphi_{n-1}(x)$$
$$(n=1) \quad \underbrace{2}_{\text{normalization}}\varphi_2(x) = 3x^2 - 1$$

Ex: $(f,g) = \int_{-1}^{1} f(x) g(x) w(x)$

$$w(x) = (1+x)^\alpha (1-x)^\beta$$
$\Rightarrow (\alpha\beta)$ Jacobi polynomials

Ex: $(f,g) = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} \Rightarrow$ Chebyshev
$(T_{n+1} = 2xT_n - T_{n-1})$

Building a general orthogonal basis.
$\hookrightarrow$ Gram-Schmidt

<u>Thm</u>: for lin. orthog. polynomials φ_n then $(p, \varphi_n) = 0 \quad \forall\ p \in P^{n-1}$

<u>Pf</u>: $p(x) = \sum_{j=0}^{n-1} c_j \varphi_j(x)$. $\quad (p, \varphi_n) = \left(\sum_{j=0}^{n-1} c_j \varphi_j, \varphi_n\right)$

$$= \sum_{j=1}^{n-1} c_j (\varphi_j, \varphi_n) = 0$$

<u>GS</u>: given any lin. basis $\{p_0, \dots, p_n\}$

$$\varphi_0(x) = p_0(x)$$

$$\varphi_1(x) = p_1(x) - \frac{(p_1, \varphi_0)}{(\varphi_0, \varphi_0)} \varphi_0(x)$$

<u>Check</u>: $(\varphi_0, \varphi_1) = \left(p_0, p_1 - \frac{(p_1, \varphi_0)}{(\varphi_0, \varphi_0)} \varphi_0\right)$

$$= (\varphi_0, p_1) - \cancel{(\varphi_0, \varphi_0)} \frac{\overbrace{(p_1, \varphi_0)}^{(\varphi_0, p_1)}}{\cancel{(\varphi_0, \varphi_0)}} = 0$$

$$\varphi_2(x) = p_2(x) - \frac{(p_2, \varphi_1)}{(\varphi_1, \varphi_1)} \varphi_1(x) - \frac{(p_2, \varphi_0)}{(\varphi_0, \varphi_0)} \varphi_0(x)$$

$\vdots$

$$\boxed{\varphi_n(x) = p_n(x) - \sum_{j=0}^{n-1} \frac{(p_n, \varphi_j)}{(\varphi_j, \varphi_j)} \varphi_j(x)}$$

Note: GS applied to monomials w/ L^2 IP
$\longrightarrow$ Legendre polynomials.

Thm: any hier. orthog. poly. admits a 3 term recurrence

$$\varphi_k(x) = p_k(x) - \frac{(p_k, \varphi_{k-1})}{(\varphi_{k-1}, \varphi_{k-1})}\varphi_{k-1}(x) - \frac{(p_k, \varphi_{k-2})}{(\varphi_{k-2}, \varphi_{k-2})}\varphi_{k-2}(x)$$

$$p_k(x) = x\varphi_{k-1}(x) \in P^k$$

Pf: Given hier. orth. $\{\varphi_0, \dots, \varphi_{n-1}\} \to$ define new hier. basis $\{\varphi_0, \dots, \varphi_{n-1}, x\varphi_{n-1}\}$ for weighted L^2 IP

$$\varphi_n(x) = x\varphi_{n-1}(x) - \sum_{j=0}^{n-1} \frac{(x\varphi_{n-1}, \varphi_j)}{(\varphi_j, \varphi_j)}\varphi_j(x)$$

$$(x\varphi_{n-1}, \varphi_j) = \int_a^b w(x)\, x\, \varphi_{n-1}(x)\varphi_j(x) = (\varphi_{n-1}, x\varphi_j)$$

$\varphi_{n-1} \in P^{n-1}$
$x\varphi_j \in P^{j+1}$ $\Rightarrow (\varphi_{n-1}, x\varphi_j) = 0$
if $j+1 < n-1$
if $j < n-2$

$$\varphi_n(x) = x\varphi_{n-1}(x) - \frac{(x\varphi_{n-1}, \varphi_{n-1})}{(\varphi_{n-1}, \varphi_{n-1})}\varphi_{n-1}(x) - \frac{(x\varphi_{n-1}, \varphi_{n-2})}{(\varphi_{n-2}, \varphi_{n-2})}\varphi_{n-2}(x)$$

L^2 IP on $[-1,1] \Rightarrow \{\varphi_i\}_{i=0}^{n}$ are Legendre polynomials

$$\Rightarrow \min_p \|f-p\|^2_{L^2([-1,1])} \Rightarrow \underbrace{G}_{\text{gram matrix}}c = b$$

$$G_{ij} = \int_{-1}^{1} \varphi_j \varphi_i = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o/w} \end{cases} \Rightarrow G = I$$

$$\Rightarrow \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} f(x)\varphi_0(x) \\ \vdots \\ \int_{-1}^{1} f(x)\varphi_n(x) \end{bmatrix}$$

In practice

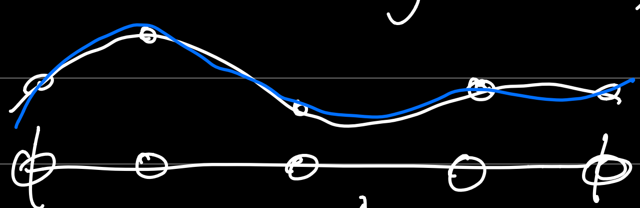
$\Rightarrow$ compute integrals numerical quadrature.

$$\int_a^b f(x) \approx \sum_{i=0}^{n} w_i f(x_i)$$

Quadrature: building L^2 Least squares

integrating non-explicit formulas

discretizing PDEs, statistical moments.



$p(x) \approx f(x)$

$$\int_a^b f(x) \approx \int_a^b p(x) = \int_a^b \sum_{i=0}^{n} f(x_i)\, l_i(x) = \sum_{i=0}^{n} f(x_i) \underbrace{\int_a^b l_i(x)}_{w_i}$$