Last time: continuous L^2 least squares

Under an orthogonal polynomial basis, the inner product matrix

$$G_{ij} = (\varphi_i, \varphi_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\Rightarrow \quad G = I.$$

Need to compute b vector with

$$b_i = (f, \varphi_i) = \int_a^b f(x)\varphi_i(x)$$

Easiest to numerically evaluate integrals using quadrature.

Suppose $p(x)$ interpolates $f(x)$, s.t.

$$p(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x) \quad (\text{Lagrange basis})$$

$$\Rightarrow \int_a^b f(x) \approx \int_a^b p(x) = \int \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

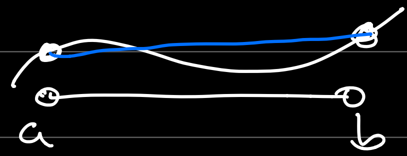
$$= \sum_{i=0}^{n} f(x_i) \underbrace{\int_a^b l_i(x)}_{w_i}$$

quadrature weights

<u>In general</u>: $\displaystyle\int_a^b f(x) = \sum_{i=0}^{n} f(x_i)\, w_i$. Will want $w_i > 0$.

<u>Newton-Cotes</u>: interpolate at $n+1$ equispaced points. Trapezoidal, Simpsons, etc...,

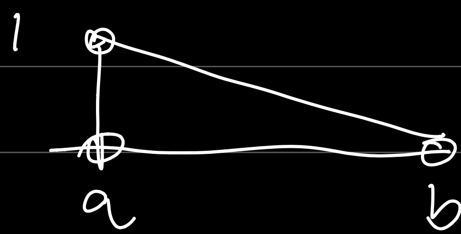
Trapezoidal rule:



$n=1$

$$p_1(x) = f(a)\, l_0(x) + f(b)\, l_1(x)$$

$$l_0(x) \to \left(\frac{b-x}{b-a}\right) \qquad l_1(x) \to \left(\frac{x-a}{b-a}\right)$$

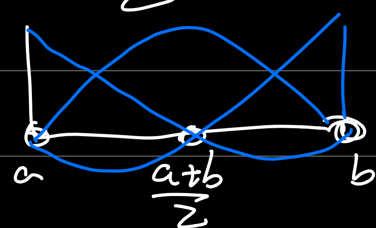


$$w_0 = \int_a^b l_0(x) = \frac{1}{2}(b-a)$$

$$w_1 = \int_a^b l_1(x) = w_0 = \frac{1}{2}(b-a)$$

$$\Rightarrow \int_a^b f(x) \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right)$$

Simpson's rule: $n=2$



$$\int_a^b f(x) \approx \sum_{i=0}^{2} f(x_i) \int_a^b l_i(x)$$

$$w_0 = w_2 = \int_a^b l_0(x) = \int_a^b l_2(x)$$

$$w_1 = \int_a^b l_1(x)$$

$$\int_a^b l_0(x) = \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \int_a^b \frac{\left(x-\frac{a+b}{2}\right)(x-b)}{\left(a-\frac{a+b}{2}\right)(a-b)}$$

after calculations $= \dfrac{b-a}{6} > 0$

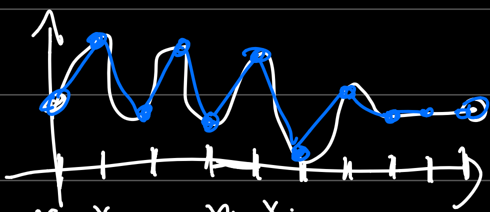
$$w_1 = \int_a^b l_1(x) = \frac{4}{6}(b-a) > 0$$

$$\int_a^b f(x) \approx \left(\frac{b-a}{6}\right)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Why do we want $w_i > 0$?

$$\int_a^b f(x) = \sum_{i=0}^{n} f(x_i) w_i \qquad f(x) = 1 \qquad \int_a^b 1 = \sum_{i=0}^{n} w_i$$

$\implies$ avoid numerical roundoff & cancellation.



Composite quadrature rule.

$$\int_a^b f(x) = \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f(x) = \sum_{i=1}^{m} \sum_{i=0}^{n} w_i f(x_i)$$

$h = \frac{(b-a)}{m}$ = size of each interval

$x_i = a + hi \qquad i = 0, \dots, m.$

$n=1$: Composite trapezoidal rule: $\sum_{i=1}^{m} \underbrace{(x_i - x_{i-1})}_{h} \frac{1}{2}(f(x_{i-1}) + f(x_i))$

$$= h\left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \dots \frac{1}{2}f(x_m)\right)$$

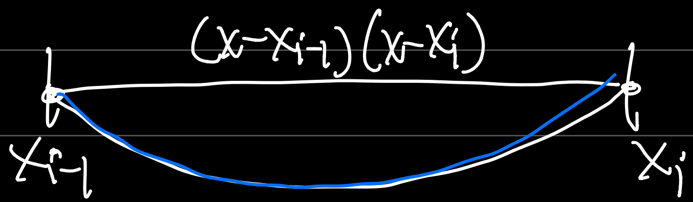
error

$$\boxed{E(f)} = \int_a^b f(x) - \sum_{i=0}^{n} w_i f(x_i) = \int_a^b (f(x) - p(x))$$

on $[x_{i-1}, x_i]$ $\quad f(x) - p(x) = \frac{f''(\eta)}{2}(x - x_{i-1})(x - x_i)$

$$E(f) = \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} (f(x) - p(x))$$

$$= \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} \frac{f''(\eta(x))}{2}(x - x_{i-1})(x - x_i)\, dx$$

$\Rightarrow$ 

MVT for integrals :

$$\boxed{\int_a^b f(x) g(x)} = \boxed{f(c) \int_a^b g(x)}$$

if $g(x) \leq 0$
or $g(x) \geq 0$

$$\Rightarrow E(f) = \sum_{i=1}^{m} \frac{f''(\eta_i)}{2} \underbrace{\int_{x_{i-1}}^{x_i} (x - x_{i-1})(x - x_i)}$$

$$\eta_i \in [x_{i-1}, x_i]$$

$$\int_{x_{i-1}}^{x_i} x^2 - (x_{i-1} + x_i) x + x_i x_{i-1}$$

$$= \frac{-1}{6} (\underbrace{x_i - x_{i-1}}_{})^3$$

$$\frac{1}{6} h^3$$

$$E(f) = h^3 \sum_{i=1}^{m} \frac{f''(\eta_i)}{12} \Rightarrow \text{use arithmetic MVT.}$$

Arith. MVT): $\frac{1}{n}\sum_{i=1}^{n} f(x_i) = f(\eta)$ where η b/w $\{x_i\}_{i=1}^{n}$

$$E(f) = \frac{h^3}{12} f''(\eta) m \qquad h = \frac{(b-a)}{m}$$

$$m = \frac{(b-a)}{h}$$

$$E(f) = (b-a)\frac{h^2}{12} f''(\eta) \quad w/ \eta \in [x_0, x_n]$$

$$|E(f)| \le \frac{(b-a)}{12} h^2 \max |f''(x)|$$

as $h \to 0$, $E(f) \propto h^2$

$$\|f-p\|_{L^\infty} \le \max_i \left|\frac{f''(\eta_i)}{2}\right| \max_i \underbrace{|(x-x_i)(x-x_{i-1})|}_{h^2/4}$$

(n=1)
<u>Trapezoidal rule</u> on $[a,b]$:

$$E(f) = -\frac{f''(\eta)}{12}(b-a)^3$$

(n=2)
<u>Simpson's rule</u>: on $[a,b]$

$$E(f) = -\frac{1}{90}\frac{(b-a)^5}{2^5} f^{(4)}(\eta)$$

<u>Polynomial exactness</u>: trapezoidal $\Rightarrow$ exact if $f(x)$ is a degree 1 polynomial.

Simpson's rule $\Rightarrow$ exact is $f(x)$ is a degree 3 poly.

Check: $\int_a^b x^3 - \underbrace{p(x)}_{n=2 \text{ interpolant}} = \frac{b^4 - a^4}{4} - \underbrace{\int_a^b p(x)}_{\frac{b-a}{6}\left(a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right)}$

$$\underbrace{\frac{b-a}{6}\,\frac{3}{2}\left(a^3 + a^2 b + a b^2 + b^3\right)}_{\frac{b-a}{4}(b+a)(b^2+a^2) \;=\; \frac{(b-a)(b^3-a^3)}{4}}$$

$\Rightarrow \quad E(x^3) = 0$

True for all Newton-Cotes quad. when $n =$ even.