Last time: composite trapezoidal rule with n points.

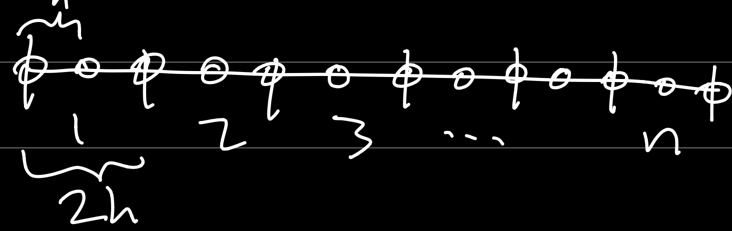
$$\int_a^b f(x) \approx h\left(\tfrac{1}{2}f(x_1) + f(x_2) + \dots f(x_{n-1}) + \tfrac{1}{2}f(x_n)\right)$$

$$h = \frac{(b-a)}{n} = \text{interval spacing.}$$

$$\Rightarrow \int_a^b f(x) - \sum_{i=1}^{n} f(x_i)\omega_i = \text{error } E(f)$$

$$E(f) = (b-a)\frac{h^2}{12} f''(\eta) \quad w/ \ \eta \in [x_0, x_n]$$

Can we do the same thing with composite Simpson's rule?



n intervals

$n + (n+1)$

$= 2n+1$

$= m$

$$\int_a^b f(x) \approx \frac{h}{3}\left(f(x_1) + 4\sum_{j=1}^{\lfloor m/2 \rfloor} f\left(a - \frac{(2j-1)}{h}\right) + 2\sum_{j=1}^{\lfloor m/2 \rfloor - 1} f(a+2jh) + f(x_m) \right)$$

$$E(f) = \frac{-h^4}{180}(b-a) f^{(4)}(\eta), \quad \eta \in [a,b]$$

$\rightarrow$ if you have a Newton-Cotes quad. with odd # of points $\Rightarrow$ it achieves one higher order accuracy, What about non-composite Newton-Cotes rules?

$x_0 \quad x_1 \quad \cdots \quad x_n$

$$\int_a^b f(x) - \underbrace{\sum_{j=0}^{n} f(x_i) w_i}_{\int_a^b p(x)} \longrightarrow w_i = \int_a^b l_i(x)$$

$$\Rightarrow w_i < 0$$

Suppose $\|f - \tilde{f}\|_{L^\infty} < \varepsilon$, $\varepsilon \ll 1$

$$\left| \sum_{i=1}^{n} f(x_i) w_i - \tilde{f}(x_i) w_i \right| \le \sum_{i=1}^{n} \underbrace{|f(x_i) - \tilde{f}(x_i)|}_{< \varepsilon} |w_i|$$

$$\le \varepsilon \sum_{i=1}^{n} |w_i| \rightarrow \text{can be large if } w_i < 0$$

1) Degree $n > 1$ composite rule = complicated
2) Newton-Cotes rules for $n \gg 1$ = unstable
3) Richardson extrapolation
 → Romberg integration

Richardson extrapolation: $x_* = \underbrace{\varphi}_{\text{numerical process}}(0)$

but we can only compute $\varphi(h)$ as an approximation w/ $h \ll 1$.

Ex: h = interval size

$$\varphi(h) = \sum_{i=1}^{n} f(x_i) w_i$$

Ex: $\varphi(h) = \frac{f(x+h) - f(x)}{h}$, $\varphi(0) \to \frac{\partial f(x)}{\delta x}$

$\hookrightarrow$ error $= O(h)$

Assume $\varphi \in C^{\infty}(\mathbb{R})$. Assume $|\varphi(h) - \varphi(0)| = O(h)$ s.t.

$$\varphi(h) = \varphi(0) + h\varphi'(0) + \frac{h^2}{2}\varphi''(0) + \cdots$$
$$= \varphi(0) + hC_1 + h^2 C_2 + \cdots$$
$$\varphi(h/2) = \varphi(0) + \frac{h}{2}C_1 + \left(\frac{h}{2}\right)^2 C_2 + \cdots$$

Define $\psi(h) = 2\varphi(h/2) - \varphi(h)$

$$= \varphi(0) - \frac{1}{2}h^2 C_2 + \cdots$$
$$|\psi(h) - \varphi(0)| = O(h^2)$$

but $|\varphi(h) - \varphi(0)| = O(h)$!

Can recurse: $\psi(h/2) = \varphi(0) - \frac{1}{2}\left(\frac{h}{2}\right)^2 C_2 + \cdots$

$$= \varphi(0) - \frac{1}{8}h^2 C_2 + \cdots$$
$$\psi(h) = \varphi(0) - \frac{1}{2}h^2 C_2 + \cdots$$

$$\frac{4\psi(h/2) - \psi(h)}{3} = \frac{3\varphi(0) + O(h^3)}{3}$$
$$= \varphi(0) + O(h^3)$$

Note: $\psi\left(\frac{h}{2}\right) = 2\varphi\left(\underline{\underline{\frac{h}{4}}}\right) - \varphi\left(\frac{h}{2}\right)$

Generalization: define R matrix

indexing from 0

$$R(j,0) = \varphi\left(\frac{h}{2^j}\right) \qquad j \text{ integer} \geq 0$$

$$R(j,k) = \frac{2^k R(j,k-1) - R(j-1,k-1)}{2^k - 1}$$

$O(h) \qquad O(h^2) \qquad O(h^3)$

$$\begin{bmatrix} \varphi(h) & & & \\ \varphi(h/2) & \psi(h) & & \\ \varphi(h/4) & \psi(h/2) & & \cdots \\ \varphi(h/8) & \psi(h/4) & & \\ \vdots & & & \end{bmatrix}$$

= assumes $|\varphi(0) - \varphi(h)| = O(h)$

If $|\varphi(0) - \varphi(h)| = h^r$

If: $\varphi(h) = \varphi(0) + C_1 h^r + C_2 h^{2r} + C_3 h^{3r} + \cdots$

Generalization of Richardson extrap.:

$$R(j,0) = \varphi(h/2^j) \qquad j \geq 0$$

$$R(j,k) = \frac{2^{rk} R(j,k-1) - R(j-1,k-1)}{2^{rk} - 1}$$

$\Rightarrow R(k,k) = O(h^{(k+1)r})$ approx to $\varphi(0)$.

Ex: Romberg integration: apply Richardson to the trapezoidal rule. $\varphi(h) = h\left(\frac{f(x_1)}{2} + f(x_2) \cdots \frac{f(x_N)}{2}\right)$

Error formula satisfies: $\varphi(h) = \int_a^b f(x) + c_1 h^2 + c_2 h^4 + \cdots$

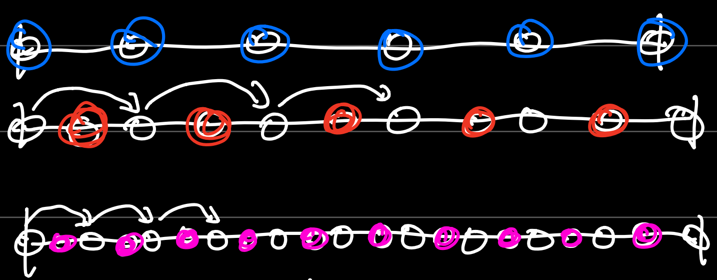
$$R(j,0) = \varphi(h/2^j)$$

$$R(j,k) = \frac{4^k R(j,k-1) - R(j-1,k-1)}{4^k - 1}$$

$\rightarrow h$

$h/2$

$h/4$



$\rightarrow$ error $\propto \frac{h^2}{16}$

$\downarrow$

error (h^8)