Interpolatory quadrature : $\int_a^b f(x) \approx \sum_{i=0}^{n} f(x_i) w_i$

with $w_i = \int_a^b l_i(x)$ , $l_i(x)$ = Lagrange polynomial

Newton-Cotes

- Trapezoidal rule: interpolate $f(x)$ with a linear polynomial $p(x)$ and approx.
  $\int_a^b f(x) \approx \int_a^b p(x) \Rightarrow$ exact if $f(x) \in P^1$
- Simpson's rule: interpolate $f(x)$ w/ quadratic
  $\Rightarrow$ exact if $f(x) \in P^3$
- Simpson's 3/8 rule: interpolate w/ cubic
  $\Rightarrow$ exact if $f \in P^3$ ??
- Boole's rule: interp. w/ quintic
  $\Rightarrow$ exact if $f \in P^5$

$\Rightarrow$ all based on equispaced points: convenient, but less accurate. How much more accurate can we get using non-equispaced points?

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Suppose $n=1$ and $\int_a^b f(x) \approx w_0 f(x_0) + w_1 f(x_1)$ with $x_0, x_1$ to be determined.

Goal: find $x_0, x_1$ such that the quadrature rule is exact for polynomials of as high a degree as possible.

Note: $x_0, x_1, w_0, w_1$ = 4 degrees of freedom

exactness for cubics $\Rightarrow$ 4 equations (constraints)

If $f \in P^3 \Rightarrow \int_{-1}^{1} f(x) = \int_{-1}^{1} a + bx + cx^2 + dx^3$

$$= a\int_{-1}^{1} 1 + b\int_{-1}^{1} x + c\int_{-1}^{1} x^2 + d\int_{-1}^{1} x^3$$

$$\sum_{i=0}^{n} f(x_i) w_i = a\sum_{i=0}^{n} 1 w_i + b\sum_{i=0}^{n} x_i w_i + c\sum_{i=0}^{n} x_i^2 w_i + d\sum_{i=0}^{n} x_i^3 w_i$$

① $w_0 + w_1 = \int_a^b 1 = b-a$

② $x_0 w_0 + x_1 w_1 = \int_a^b x$

③ $x_0^2 w_0 + x_1^2 w_1 = \int_a^b x^2$

④ $x_0^3 w_0 + x_1^3 w_1 = \int_a^b x^3$

nonlinear equations w.r.t. $x_0, x_1, w_0, w_1$

$\Rightarrow$ $w_0 = w_1 = \frac{1}{2}(b-a)$

$x_0 = \frac{1}{2}(b+a) - \frac{\sqrt{3}}{6}(b-a)$

$x_1 = \frac{1}{2}(b+a) + \frac{\sqrt{3}}{6}(b-a)$

$x_0, x_1 \in [a, b]$

Generalize beyond 2 points: $n+1$ pts

$2n+2$ points + weights $\Rightarrow$ expect to integrate deg. $2n+1$ polynomials exactly

( $\sum_{j=0}^{n} x_j^i w_j = \int_a^b x^i$ for $i = 0, \ldots, 2n+1$ exactness conditions )

$\Rightarrow$ look for nodes $x_0, \ldots, x_n$

determine $w_i = \int_a^b l_i(x)$

Hierarchical orthog. polynomials connected to "optimal" nodes $x_0, \dots, x_n$.

Recall $\{\varphi_i\}_{i=0}^{n}$ which are orthog. w.r.t. $L^2$

$$(\varphi_i, \varphi_j) = \int_a^b \varphi_i(x)\varphi_j(x) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and $\varphi_i$ is a polynom. exactly of deg $i$.

**Euclidean division** - polynom. long division

Euclidean division of polynomials, which is used in Euclid's algorithm for computing GCDs, is very similar to Euclidean division of integers. Its existence is based on the following theorem: Given two univariate polynomials $a$ and $b \neq 0$ defined over a field, there exist two polynomials $q$ (the *quotient*) and $r$ (the *remainder*) which satisfy

$$a = bq + r$$

and

$$\deg(r) < \deg(b),$$

where "deg(...)" denotes the degree and the degree of the zero polynomial is defined as being negative. Moreover, $q$ and $r$ are uniquely defined by these relations.

$p \in P^{2n+1}$. There exists a representation such that

$$p(x) = \varphi_{n+1}(x)\, q(x) + r(x)$$

w/ $q, r \in P^n$

$\varphi_{n+1} \in P^{n+1}$

$$\int_a^b p(x) = \overbrace{\cancel{\int_a^b \underbrace{\varphi_{n+1}(x)}_{\in P^{n+1}} \underbrace{q(x)}_{\in P^n}}}^{0} + \int_a^b r(x)$$

$$\sum_{i=0}^{n} p(x_i)\, w_i = \underbrace{\sum_{i=0}^{n} \varphi_{n+1}(x_i)\, q(x_i)\, w_i}_{\text{choose } x_i \text{ st this} = 0} + \sum_{i=0}^{n} \overbrace{r(x_i)}^{\in P^n} w_i$$

$\Rightarrow$ choose $x_i$ s.t. $\varphi_{n+1}(x_i) = 0$

$\Rightarrow$ exactly $n+1$ roots to this polynom.

If $x_i$ are roots of $\varphi_{n+1}$, $\sum_{i=0}^{n} w_i f(x_i)$ is exact for $f \in P^{2n+1}$.

Need <u>distinct</u> roots of $\varphi_{n+1}(x)$!

<u>Thm</u>: $\varphi_k$ is a degree $k$ orthog. polynom. w.r.t. a weight $w(x) > 0$ on $(a,b)$, then $\varphi_k$ has $k$ $\underbrace{\text{distinct}}$ roots $x_j \in [a,b]$.

($x_i$ distinct if $x_i \neq x_j$ for $j \neq i$ and multiplicity $= 1$)

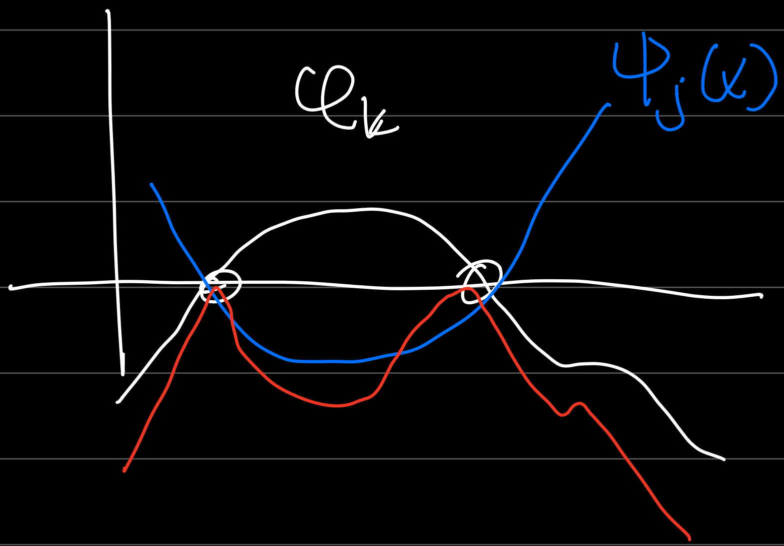
<u>Pf</u>: By contradiction. Suppose $\varphi_k$ has $j < k$ distinct roots $x_j \Rightarrow \varphi_k$ changes sign at $j < k$ distinct points.

Define $\psi_j(x) = (x-x_1)(x-x_2) \cdots (x-x_j) \in P^j$

$\varphi_k(x)\psi_j(x) =$ has same sign everywhere in $[a,b]$

$\int_a^b \varphi_k(x)\psi_j(x) > 0$ or $< 0$ except at $j$ points. but $\varphi_k$ is orthog to $P^j$ $\forall j < k$

$\Rightarrow \int_a^b \varphi_k(x)\psi_j(x) = 0 \Longrightarrow$ contradiction.




$\Longrightarrow$ Illustration of $\varphi_k, \psi_j$

Gaussian quadratures : $n+1$ roots of an orthog. polynomial $\varphi_{n+1}(x)$. What about weights?

Want $w_i > 0$

$\hookrightarrow w_i = \int_a^b l_i(x)\, \underbrace{w(x)}_{\text{weight for an inner prod.}} > 0$

$\rightarrow$ Show on hw.

$\Rightarrow$ $n+1$ point quad. rule which is exact for all $p \in P^{2n+1}$

$n+1 = 2 \Rightarrow p \in P^1$ Trapezoidal

$n+1 = 3 \Rightarrow p \in P^3$ Simpsons

$n+1 = 3 \Rightarrow p \in P^5$ Gaussian