Rootfinding: solving equations numerically.

$\Rightarrow$ Given $f(x)$, find x s.t. $f(x) = 0$

Ex: $e^{ax} = b \Rightarrow \underbrace{e^{ax} - b}^{f(x)} = 0 \Rightarrow$ $\log(e^{ax}) = \log(b)$

$x = \frac{\log(b)}{a}$

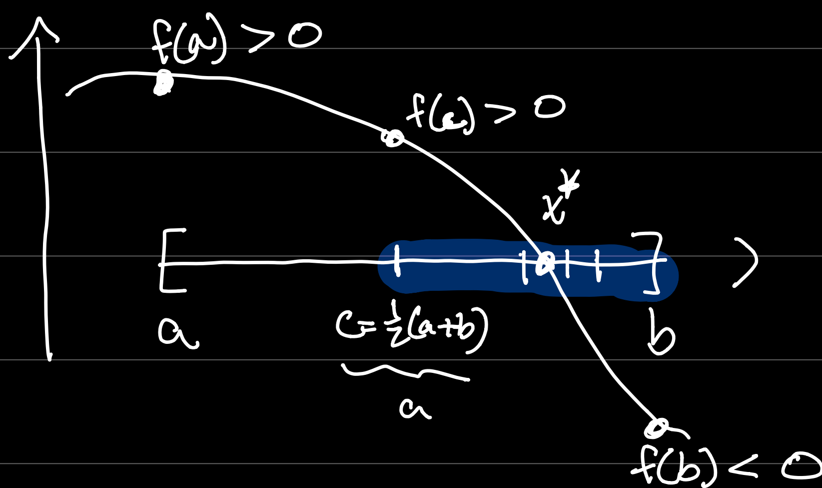
For more complex $f(x)$, need a numerical method to approximate $x \Rightarrow$ rootfinding method

What you should learn today:

- how to implement + analyze the bisection method
- Newton's method and how it compares to bisection

Simplest method: bisection.

- Let $f(x): \mathbb{R} \to \mathbb{R}$ on $[a,b]$ (set by user)
- Assume that $\exists$ a root x^* s.t. $f(x^*) = 0$
  & $x^* \in [a,b]$ & f is continuous.



Algorithm: given $[a,b]$, init $c = \frac{1}{2}(a+b)$

while $|f(c)| > tol$

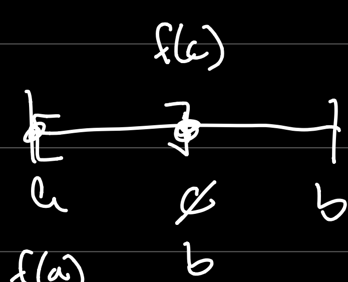
if $f(c) f(a) < 0$

  set $b = c$

else

  set $a = c$

$c = \frac{1}{2}(a+b)$

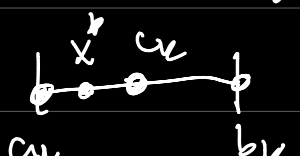


<u>Analysis</u>: prove that bisection works

e.g., $c_k \longrightarrow x^*$

1) by construction $\rightarrow [a_k, b_k]$ at kth iteration, & $x^* \in [a_k, b_k]$

2) by kth iteration $c_k = \frac{1}{2}(a_k + b_k)$



$$|x^* - c_k| = \frac{1}{2}|b_k - a_k|$$

$$(b_k > a_k) \quad = \frac{1}{2}\frac{1}{2}|b_{k-1} - a_{k-1}|$$

$$= \left(\frac{1}{2}\right)^k |b - a|$$

$\Rightarrow$ converge as 2^{-k}

<u>Advantage</u>: simple $\longrightarrow$ robust

easy to code

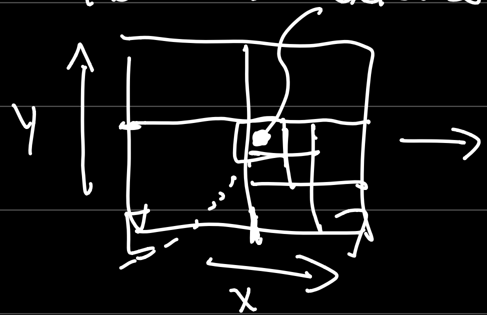
<u>Disadvantages</u>: relatively slow

$\Rightarrow$ $tol = 10^{-12} = \left(\frac{1}{2}\right)^k \Rightarrow k = \log_2(10^{12})$

$\Rightarrow k = 40$ iterations.

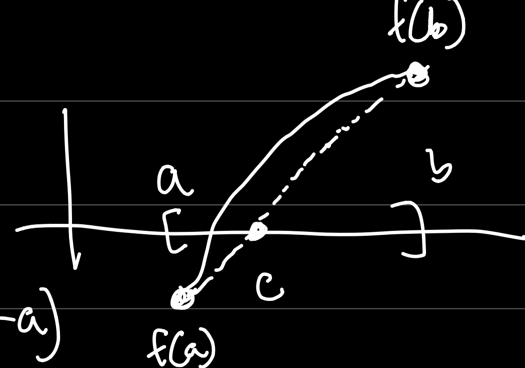
Other rootfinders $\Rightarrow k = 5$ iterations.

— Hard to extend to higher dimensions



→ region trees

— Regula falsi :

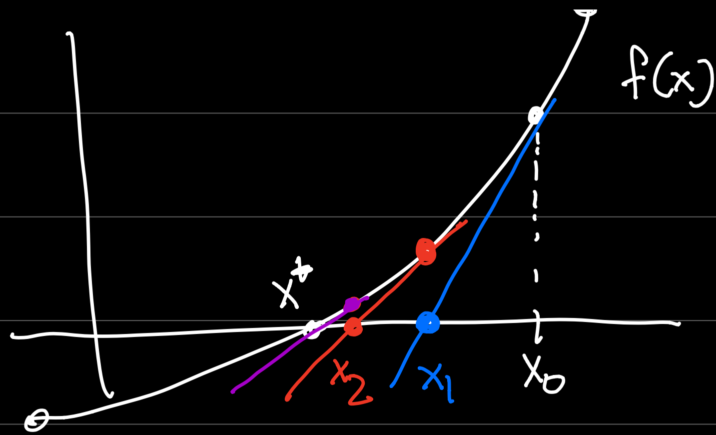


$$p(x) = f(a) + \frac{(f(b) - f(a))}{b-a}(x-a)$$

$$\hookrightarrow \quad x = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

→ unreliable

Newton:



repeat

$|f(x_k)| < tol$

Algorithm:

Assume x_0, $f(x)$, $f'(x)$

while $|f(x_k)| > tol$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Motivate:

$$f(x_*) = 0 = f(x_k) + f'(x_k)(x_* - x_k) + \cancel{\frac{f''(\eta)}{2}(x_* - x_k)^2}$$

ignore

(Assume $f \in C^2(\mathbb{R})$)

$$x_k - \frac{f(x_k)}{f'(x_k)} = x_*$$

Convergence of Newton: why & when?

- define assumptions carefully
- Asymptotic convergence

$\Rightarrow$ as $k \to \infty$

failure
- Can oscillate (cycling)
- can diverge
- $x_k - \frac{f(x_k)}{f'(x_k)}$ $\Rightarrow$ can divide by zero.