Last time: given $\int_a^b f(x)$, Gaussian quadrature points are defined as roots of $\varphi_{n+1}(x)$ = degree $n+1$ orthogonal polynomial w.r.t. the L^2 inner product.

$$\Rightarrow \quad \int_a^b f(x) = \sum_{i=0}^{n} w_i f(x_i) \quad \text{for all } f \in P^{2n+1}$$

How to analyze? For other interpolatory quadrature,

$$\sum_{i=0}^{n} w_i f(x_i) = \int_a^b p(x), \quad \text{where } p(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$$
$$w_i = \int_a^b l_i(x).$$

However, this only gives exactness for degree n (or $n+1$) integrands, and Gaussian quadrature is exact for degree $2n+1$ polynomials.

What type of $n+1$ point interpolant produces a degree $2n+1$ polynomial? Hermite!

Hermite polynom. interp.: $$P_{2n+1}(x) = \sum_{k=0}^{n} f(x_k) A_k(x) + \sum_{k=0}^{n} f'(x_k) B_k(x)$$

w/ $$A_k(x) = l_k^2(x)\left(1 - 2 l_k'(x_k)(x - x_k)\right)$$
$$B_k(x) = l_k^2(x)(x - x_k)$$

$$\Rightarrow \quad A_k(x_j) = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{o/w} \end{cases}$$
$$A_k'(x_j) = 0$$

vice versa for $B_k(x_j)$

Assuming that Gaussian quadrature does correspond to Hermite interpolation ...

$$\int_a^b f(x) \approx \int_a^b P_{2n+1}(x) = \int_a^b \sum_{k=0}^{n} f(x_k) A_k(x) + \sum_{k=0}^{n} f'(x_k) B_k(x)$$

$$= \sum_{k=0}^{n} f(x_k) \int_a^b A_k(x) + \sum_{k=0}^{n} f'(x_k) \int_a^b B_k(x)$$

$$\overset{???}{=} \sum_{k=0}^{n} f(x_k) \omega_k . \quad \text{Want to show } \int_a^b B_k(x) = 0$$

$$\int_a^b B_k(x) = \int_a^b l_k^2(x)(x - x_k)$$

$$(x-x_k)\, l_k(x) = \left(\prod_{j \neq k} \frac{(x - x_j)}{(x_k - x_j)} \right)(x - x_k) = \frac{\prod_{j=0}^{n} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

$$l_k(x)(x - x_k) = \underbrace{C_n}_{1/\prod_{j \neq k}(x_k - x_j)} \underbrace{\Pi_{n+1}(x)}_{\prod_{j=0}^{n}(x - x_j)}$$

$\Pi_{n+1}(x)$ has roots at x_i & is degree $n+1$.

$\Rightarrow$ x_i roots of orthog. polynom. $\varphi_{n+1}(x)$.

$$\Pi_{n+1}(x) = \text{scaling of } \varphi_{n+1}(x)$$

$$\int_a^b B_k(x) = \int_a^b \underbrace{l_k(x)}_{\in P^n} \underbrace{l_k(x)(x - x_k)}_{C_n \Pi_{n+1}(x) = \text{scaling of } \varphi_{n+1}(x)} = 0$$

What about $\int_a^b A_k(x)$? $\quad A_k(x) = l_k^2(x)\left(1-2l_k'(x_k)(x-x_k)\right)$

$$w_k = \int_a^b A_k(x) = \underbrace{\int_a^b l_k^2(x)}_{>0} - 2l_k'(x_k)\underbrace{\int_a^b l_k^2(x)(x-x_k)}_{0=\int_a^b C_n T_{n+1}(x)\, l_k(x)} \to 0$$

$\Rightarrow$ all weights positive!

simpler

$$\int_a^b l_k^2(x) = \sum_{j=0}^{n} \underbrace{l_k^2(x_j)}_{=0 \text{ if } j\neq k} w_j = w_k$$

(but not helpful for error analysis)

$$\int_a^b f(x) = \int_a^b p_{2n+1} \Rightarrow E(f) = \int_a^b f(x) - p_{2n+1}(x)$$

Error estimate: Let $f \in C^{2n+2}[a,b]$. $\exists \eta \in [a,b]$ s.t.

$$\int_a^b f(x) - \sum_{i=0}^{n} f(x_i) w_i$$

$$= \frac{f^{(2n+2)}(\eta)}{(2n+2)!} \int_a^b \psi^2(x)$$

w/ $\psi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

Pf: $\int_a^b f(x) - p_{2n+1}(x)$

$$= \int_a^b \frac{f^{(2n+2)}(\xi(x))}{(2n+2)!} \psi^2(x)\, dx$$

MVT: If $g(x)$ doesn't change sign, $\exists \eta$ s.t. $\int_a^b f(x) g(x) = f(\eta) \int_a^b g(x)$

$$\Rightarrow \int_a^b \frac{f^{(2n+2)}(\xi(x))}{(2n+2)!} \psi^2(x) = \frac{f^{(2n+2)}(\eta)}{(2n+2)!} \int_a^b \psi^2(x) \quad \square$$

Gauss quadrature cousins

Gauss-Legendre – standard Gauss quad.

Gauss – Jacobi – $\int_{-1}^{1} f(x)\, w(x)$ w/ $w(x) = (1-x)^{\alpha}(1-x)^{\beta}$

Gauss – Chebyshev – $w(x) = 1/\sqrt{1-x^2}$

Gauss – Radau – on $[-1,1]$, Left / right endpoint $= \pm 1$

Gauss – Lobatto – on $[-1,1]$, quad. pt's include both ± 1

Gauss – Laguerre – infinite integral

$\vdots$

→ Composite Gauss quad. rules.