Next section: numerical linear algebra.
Assumes basic linear algebra knowledge, but focuses on numerical tools. Matrix factorizations.

Suppose $A \in \mathbb{R}^{m \times n}$ is a matrix.
$x \in \mathbb{R}^n$

Norms: norms on vectors $\|x\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}$
Common choices for $p = 1, 2, \infty$
w/ $\|x\|_\infty = \max_i |x_i|$

Norms on matrices: definitions

1. $\|A\| \geq 0$ & $\|A\| = 0 \iff A = 0$
2. $\|\alpha A\| = |\alpha| \, \|A\|$
3. $\|A + B\| \leq \|A\| + \|B\|$

Induced norms on matrices:

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}, \quad p = 1, 2, \infty \text{ common}$$

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |A_{ij}|$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |A_{ij}|$$

$m = n$
(for square matrices)

$$\|A\|_2 = \max_i |\lambda_i|, \quad \lambda_i = \text{eigenvalue of } A.$$

Non-induced norms: $\|A\|_F^2 = \sum_i \sum_j |A_{ij}|^2$

Cauchy-Schwarz inequality:

assuming $B \neq 0$

$$\|AB\| = \max_{x\neq 0} \frac{\|ABx\|}{\|x\|} = \max_{x\neq 0} \frac{\|ABx\|}{\|Bx\|} \frac{\|Bx\|}{\|x\|}$$

$y = Bx$

$$\leq \max_{y\neq 0} \frac{\|Ay\|}{\|y\|} \max_{x\neq 0} \frac{\|Bx\|}{\|x\|}$$

$$\leq \|A\| \quad \|B\|$$

& holds for Frobenius norm.

Vector space settings: $U \subset \mathbb{C}^n$

U is a subspace if $u, v \in U \Rightarrow \alpha u + \beta v \in U$

α, β = scalars

$$R(A) = \mathrm{Ran}(A) = \{Ax : x \in \mathbb{C}^n\}$$

$$N(A) = \mathrm{Ker}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$$

Rank of A = $\dim(R(A))$ = # of basis vectors needed to represent $R(A)$

Some Matrix A is nonsingular if

① $R(A) = \mathbb{C}^n \Longleftrightarrow \mathrm{rank}(A) = n$

② $N(A) = \{0\}$

If A is square $\left\{ \begin{array}{l} \text{③ } A \text{ has no zero eigenvalues} \\ \text{④ } A^{-1} \text{ exists \& is well defined} \end{array} \right.$

(Conj) transpose : $x^* = \bar{x}^T$. $A^* = \bar{A}^T$

Orthogonality : $x, y \in \mathbb{C}^n$ are orthogonal if

$$x^* y = 0 \Rightarrow x \perp y$$

Subspaces U, V are orthog. if $u \perp v$ for all $u \in U$, $v \in V$

Note : $y \in R(A)$, $x \in N(A^*)$

$y = Az$ for some z $\quad$ $A^* x = 0$

$$x^* y = x^* A z = (A^* x)^* z = 0$$

Fundamental Thm of linear algebra : for $A \in \mathbb{C}^{m \times n}$

$$R(A) \perp N(A^*)$$
$$R(A^*) \perp N(A)$$
$$\mathbb{C}^m = R(A) \oplus N(A^*)$$
$$\mathbb{C}^n = R(A^*) \oplus N(A)$$

QR factorization $\Rightarrow$ $A = QR$

$$\begin{array}{ccc} A & = Q & R \\ m\times n & m\times m & m\times n \\ & m\times n & n\times n \\ & \downarrow & \downarrow \\ & \text{"unitary"} & \text{upper triangular} \\ & Q^*Q = I & \end{array}$$

$\downarrow$

Projector and reflector matrices.

$P \in \mathbb{C}^{n\times n}$ is a projector if $P^2 = P$ (idempotent)

$$P^2 x \in R(P)$$

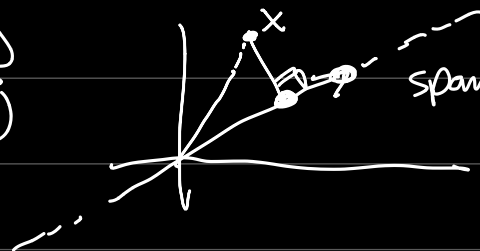
$P = P^* \Rightarrow$ orthogonal projector,

$$x - Px \perp R(P)$$

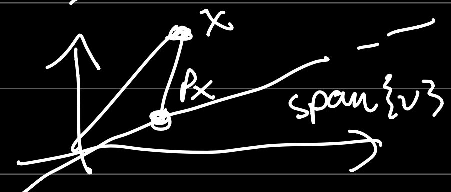
(by FTLA) $\Rightarrow x - Px \in N(P^*) = N(P)$

Ex: Given $v \in \mathbb{C}^n$, $P = \frac{vv^*}{v^*v}$ = orthogonal projector

$$Px = v\frac{(v^*x)}{(v^*v)}$$



Ex: $P = \frac{uu^*}{v^*u}$



Complementary projectors: If $P^2 = P$, $Q = I - P$ is a projector onto $N(P) \Rightarrow PQ = 0$

$$(PQ = P(I-P) = P - P^2 = 0)$$

$$R(Q) = N(P)$$
$$R(P) = N(Q)$$

Householder reflectors: built from projectors & complementary projectors.

$v \in \mathbb{C}^n$

$$H(v) = I - 2P, \quad P = \frac{vv^*}{v^*v}$$

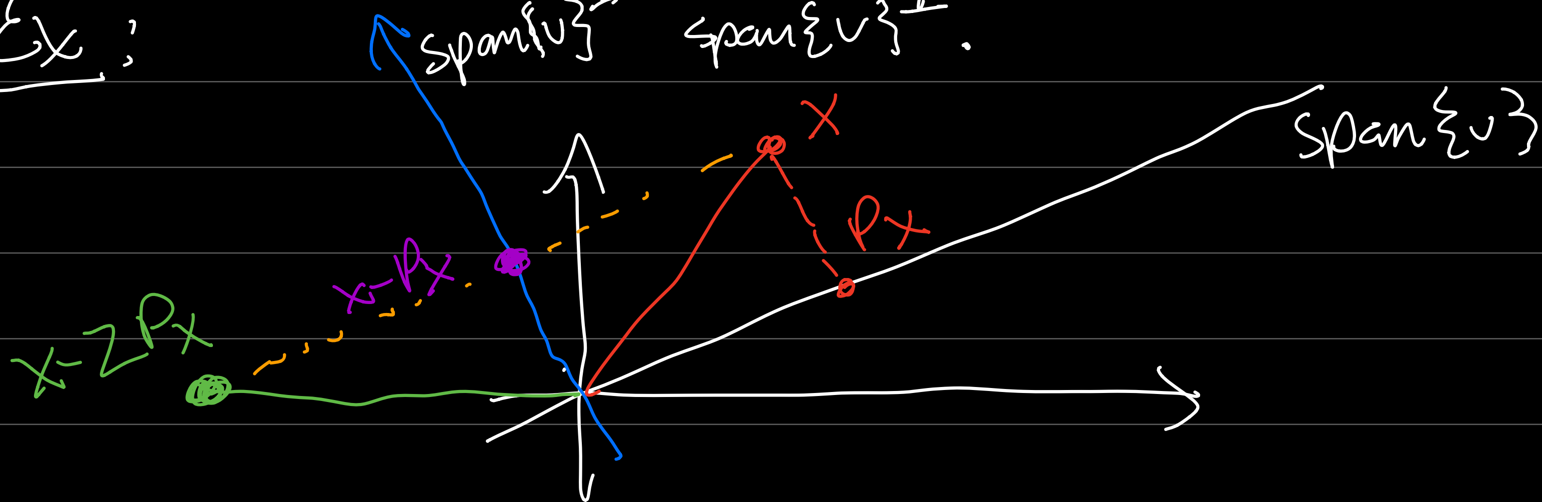
$H(v)$ Hermitian & unitary $\quad H(v)^* H(v) = I$

$\qquad\qquad\qquad\qquad\qquad\quad H(v) H(v)^* = I$

$$H(v)x = \underbrace{x - 2Px}_{}$$

reflection of x over the $n-1$ dimension hyperplane defined by $\operatorname{span}\{v\}^\perp$.

Ex:



Unitary matrix $U^*U = I$

$$\Rightarrow \|Ux\|^2 = (Ux)^*Ux = x^*U^*Ux$$
$$= x^*x = \|x\|^2$$

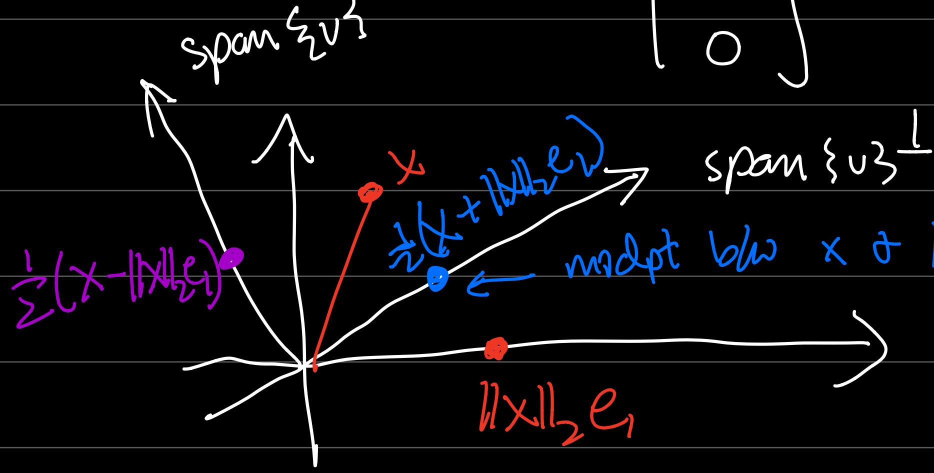
$\Rightarrow$ $H(v)x$ doesn't change norm.

Use $H(v)$ to zero out entries of a vector using unitary matrix operations

Goal: given x, find v s.t.

$$H(v)\,x = \begin{bmatrix} \|x\|_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \|x\|_2 e_1$$

$$\left(e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$



$$v = x - \|x\|_2 e_1 \implies H(v)\,x = \|x\|_2 e_1$$