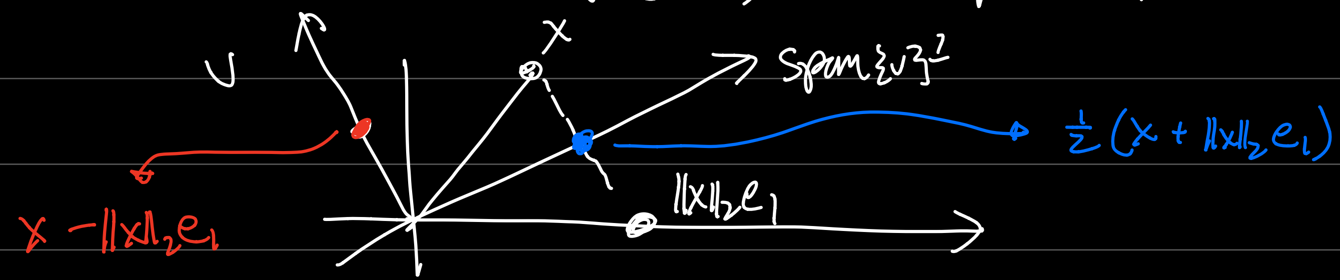
Last time: $H(v(x))\,x = \|x\|_2 e_1$



$$H(v) = I - 2\frac{vv^*}{v^*v} \implies H(v)x = \|x\|_2 e_1$$

<u>Matrix factorizations:</u> $A \in \mathbb{R}^{m\times n}$, $A = QR$

Q = unitary, R = upper triangular

Let $A = \left[\, a_1 \;\vdots\; \hat{A}_1 \,\right]$. $\quad v_1 = a_1 - \|a_1\| e_1$

$$\Rightarrow H(v_1)A = \left[\, H(v_1)a_1 \;\vdots\; H(v_1)\hat{A}_1 \,\right]$$

$$= \left[\begin{array}{c|c} \|a\|_2 & \\ 0 & H(v_1)\hat{A}_1 \\ \vdots & \\ 0 & \end{array}\right]$$

relabeling entries $= \left[\begin{array}{c|cccc} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ \hline 0 & & & & \\ \vdots & & A_2 & & \\ 0 & & & & \end{array}\right]$

$A_2 = \left[\, a_2 \;\vdots\; \hat{A}_2 \,\right]$ repeat same steps

$\Rightarrow$ def. $v_2 = a_2 - \|a_2\| e_1 \quad (e_1 \in \mathbb{R}^{m-1})$

$$\Rightarrow \quad H(v_2)\,a_2 = \|a_2\| e_1$$

Must apply $H(v_2)$ to zero out column a_2 of $\hat{A}_2$

$\Rightarrow$ cast as a matrix operation on A.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & H(v_2) \end{bmatrix}}_{Q_2} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & & A_2 & \end{bmatrix}}_{\underbrace{H(v_1)}_{Q_1} A} = \begin{bmatrix} r_{11} & r_{12} & \cdots & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} \\ 0 & 0 & & & \\ \vdots & \vdots & & A_3 & \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} I_{2\times 2} & \\ & H(v_3) \end{bmatrix}}_{Q_3} \quad Q_2\, Q_1\, A = \begin{bmatrix} r_{11} & r_{12} & \cdots & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & \cdots & r_{2n} \\ \vdots & 0 & r_{33} & \cdots & r_{3n} \\ 0 & \vdots & & & \\ & 0 & & A_4 & \end{bmatrix}$$

$$\Rightarrow \underbrace{Q_{n-1}}_{\text{unitary}} Q_{n-2} \cdots \underbrace{Q_1}_{\text{unitary}} A = \underbrace{\begin{array}{c|c} & n \\ m & \begin{bmatrix} r_{11} & \cdots & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ 0 & & r_{33} \cdots & r_{3n} \\ & & \ddots & \vdots \\ & & & r_{nn} \\ \hline & & 0 & \\ & \longleftarrow & n & \longrightarrow \end{bmatrix} \end{array} \begin{array}{l} \\ n \\ \\ \\ \\ \uparrow \\ m-n \\ \downarrow \end{array}}_{R}$$

$$Q_1^* \cdots Q_{n-1}^* \overbrace{Q_n^* \, Q_n}^{I} \, Q_{n-1} \cdots Q_1 A$$

$$\Rightarrow \quad A = Q_1^* \cdots Q_{n-1}^* R$$

Q

If $A \in \mathbb{R}^{m\times n}$ $\quad m \le n$

$$Q \in \mathbb{R}^{m\times m}, \quad R \in \mathbb{R}^{m\times n}$$

$$\Rightarrow \; {}_{m}\Big[\; \underset{m}{Q} \;\Big]_{m} \underbrace{\begin{bmatrix} \overset{m}{\diagdown} & \big| & \\ 0 & \big| & \end{bmatrix}}_{R}$$

<u>Thm</u>: For any $A \in \mathbb{R}^{m\times n}$, $\exists$ $Q \in \mathbb{R}^{m\times m}$ unitary & $R \in \mathbb{R}^{m\times n}$ upper triangular s.t. $A = QR$.

$$Q_1^* \cdots Q_i \cdots Q_n^*$$

$Q_1^* \downarrow \underbrace{\begin{bmatrix} H(v_1) \end{bmatrix}}_{\text{dense}}$ $\quad Q_i \downarrow \begin{bmatrix} I & \\ & \underbrace{H(v_i)}_{\text{dense}} \end{bmatrix}$

$\Rightarrow$ don't usually want to explicitly construct Q.

$\Rightarrow$ can instead store $v_1, \dots, v_n$ (Householder vectors) and just apply Q to some vector

$$Qx = Q_1^* \cdots \underbrace{Q_n^*}_{} x$$

$$\begin{bmatrix} I & \\ & H(v_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ H(v_n) x_2 \end{bmatrix}$$

$H(v_n)x_2 = x_2 - 2 v_n \dfrac{v_n^* x_2}{v_n^* v_n}$ = matrix-free operations.

Complexity $\approx 2mn^2 - \dfrac{2n^3}{3}$ operations

$= O(n^3)$ for A square.

Uses of Householder QR:

- if you want an orthonormal basis for $\mathbb{R}^m$

$\Rightarrow$ get full matrix $Q \in \mathbb{R}^{m\times m}$

$$\min \|Ax - b\|_2^2 \Rightarrow A^*Ax = A^*b$$

$$(QR)^*(QRx) = (QR)^*b$$

If R invertible (e.g. A full column rank)

$$\cancel{R^*}Rx = \cancel{R^*}Q^*b$$

$$Rx = Q^*b$$

$$A = \begin{array}{c} n \\ \begin{bmatrix} \quad \end{bmatrix} m \end{array}$$

$$\begin{array}{c} n \end{array} \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix}^{n} = \begin{bmatrix} - q_1^* - \\ \vdots \\ - q_m^* - \end{bmatrix} b$$

$$\hat{R}x = \underbrace{\begin{bmatrix} - q_1^* - \\ \vdots \\ - q_n^* - \end{bmatrix}}_{\text{only need } n \text{ columns.}} b$$

don't need $q_{n+1}, \dots, q_m$!

"Skinny" QR factorization of $A \in \mathbb{R}^{m\times n}$ $m \geq n$

$A = QR$ but $Q \in \mathbb{R}^{m\times n}$
$R \in \mathbb{R}^{n\times n}$

Gram-Schmidt: $A = \begin{array}{c} n \\ \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} m \end{array}$, orthogonalizes columns a_j

$$q_1 = a_1 / \|a_1\|$$
$$\hat{q}_2 = a_2 - q_1 (q_1^* a_2)$$
$$q_2 = \hat{q}_2 / \|\hat{q}_2\|$$
$$\vdots$$
$$\left.\begin{aligned} \hat{q}_k &= a_k - \sum_{j=1}^{k-1} (q_j^* a_k) q_j \\ \rightarrow q_k &= \hat{q}_k / \|\hat{q}_k\| \end{aligned}\right\}$$ hides a QR factorization

$$\|\hat{q}_k\| q_k + \sum_{j=1}^{k-1} (q_j^* a_k) q_j = a_k$$

$$\begin{bmatrix} | & & | \\ q_1 & \cdots & q_k \\ | & & | \end{bmatrix} \begin{bmatrix} (q_1^* a_k) \\ (q_2^* a_k) \\ \vdots \\ \|\hat{q}_k\| \end{bmatrix} = \begin{bmatrix} | \\ a_k \\ | \end{bmatrix}$$

$$\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} \|\hat{q}_1\| & (q_1^* a_2) & (q_1^* a_3) & \cdots \\ & \|\hat{q}_2\| & (q_2^* a_3) & \cdots \\ & & \ddots & \vdots \\ 0 & & & \end{bmatrix} = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$$

$$Q \qquad R = A$$