Last time: SVD of a full rank matrix $A \in \mathbb{C}^{m\times n}$, $m \geq n$

① Let $v_1, \dots, v_n$; $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n > 0$ $\Big\}$ = orthonormed eigenvectors & eigenvalues of A^*A.

② Define $\sigma_j = \| A v_j \| = \sqrt{v_j^* A_j^* A_j v_j} = \sqrt{\lambda_j} > 0$

③ Define $u_j = \frac{1}{\sigma_j} A v_j$ (note $u_j^* u_k = \frac{1}{\sigma_j \sigma_k} v_j^* A^* A v_k = 0$)

$\sigma_j u_j = A v_j$ for $j = 1, \dots, n$ implies

$$\underbrace{{}_{m}\begin{bmatrix} | & & | \\ u_1 & \dots & u_n \\ | & & | \end{bmatrix}}_{n} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}_{n}^{n} = A \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}_{n}^{n}$$

$$\hat{U}\hat{\Sigma} = AV \quad (\text{by } V \text{ unitary})$$

$\Rightarrow A = \hat{U}\hat{\Sigma}V^*$ "reduced" SVD

$$m\underbrace{\begin{bmatrix} A \end{bmatrix}}_{n} = m\underbrace{\begin{bmatrix} \hat{U} \end{bmatrix}}_{n} {}_{n}\underbrace{\begin{bmatrix} \hat{\Sigma} \end{bmatrix}}_{n} \underbrace{\begin{bmatrix} V^* \end{bmatrix}}_{n}{}_{n}$$

$$m\underbrace{\begin{bmatrix} A \end{bmatrix}}_{n} = m\overbrace{\begin{bmatrix} \; U \; \end{bmatrix}}^{m} \overbrace{\begin{bmatrix} \Sigma \end{bmatrix}}^{n}_{m} \overbrace{\begin{bmatrix} V^* \end{bmatrix}}^{n}{}_{n}$$

"full" SVD
U, V unitary.

If $m > n$, A full rank: $\hat{U}\,\hat{\Sigma} \to U\Sigma$

$$\Sigma = \begin{bmatrix} \hat{\Sigma} \\ 0 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & 0 & \end{bmatrix} \end{array} \begin{array}{l} n \\ m-n \end{array}$$

then $\hat{U}\hat{\Sigma} = U\Sigma$ for any $U = \left[\hat{U} \middle| u_{n+1} \cdots u_m\right]$

Want $u_{n+1}, \dots, u_m \perp u_i$ $i = 1, \dots, n$

$\Rightarrow$ $U^* U = I$ if $\|u_i\| = 1$

Can construct $u_{n+1}, \dots, u_m$ via Gram-Schmidt.

If $m \geq n$, $\mathrm{rank}(A) \leq n$:

① Let $v_1, \dots, v_n$
$\lambda_1 \geq \lambda_2 \geq \dots \lambda_n \geq 0$ $\Big\}$ = orthonormed eigenvectors & eigenvalues of A^*A.

② Define $\sigma_j = \|Av_j\|$ $j = 1, \dots, n$

③ Define $u_j = \frac{1}{\sigma_j} Av_j$ $j = 1, \dots, \underbrace{\mathrm{rank}(A)}_{r = \mathrm{rank}(A)}$

(ensures $\sigma_j > 0$)

Then we have $\{u_1, \dots, u_r\}$

$$A = \underbrace{\begin{bmatrix} u_1 & \cdots & u_r & \bigg| & u_{r+1} & \cdots & u_m \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & 0 & \end{bmatrix}}_{\Sigma} V^*$$

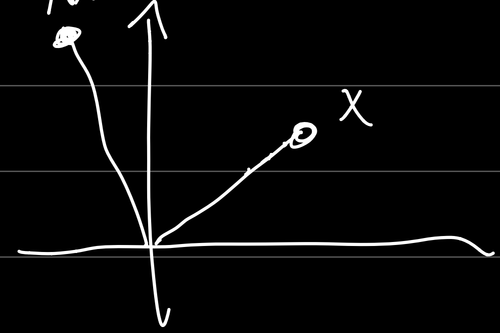
If we have $A \in \mathbb{C}^{m \times n}$ $m \leq n$: apply SVD to A^*

$A^* = U\Sigma V^*$

$\Rightarrow$ $A = V\Sigma^T U^*$ is an SVD

Ax

(2D) Geometric interp:



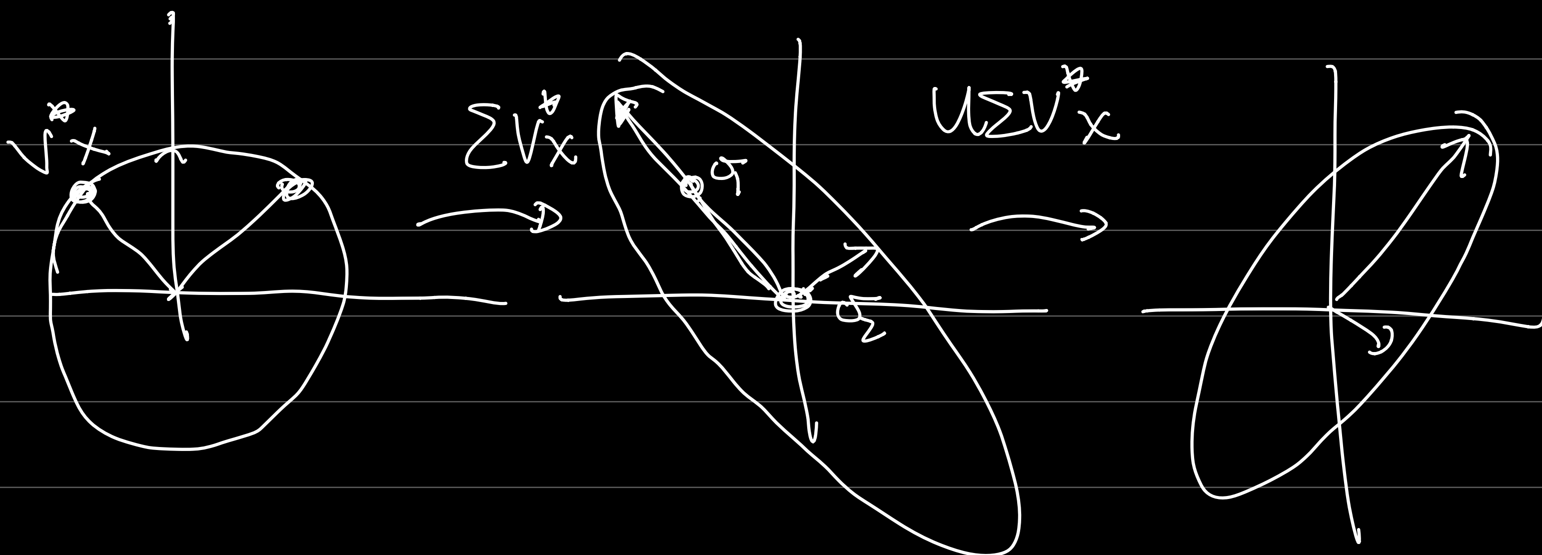
Ax

$$Ax = U\Sigma V^* x$$

U, V unitary $\Rightarrow$ $V^* x$ rotation / change of basis

$$\Sigma y = \begin{bmatrix}\sigma_1 & \\ & \sigma_2\end{bmatrix}\begin{pmatrix}y_1 \\ y_2\end{pmatrix} = \begin{bmatrix}\sigma_1 y_1 \\ \sigma_2 y_2\end{bmatrix} \quad \text{scaling}$$

Uw = rotation / change of basis.



Relationship to 2-norm of matrices:

Recall $\|A\|_2 = \max_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$

$$= \max_{x \neq 0} \frac{\|U\Sigma V^* x\|}{\|x\|} \qquad \|Ux\| = \|x\| \text{ if } U \text{ unitary}$$

$$= \max_{x \neq 0} \frac{\|\Sigma V^* x\|}{\|x\|}$$

$$= \max_{x \neq 0} \frac{\|\Sigma V^* x\|}{\|V^* x\|} = \max_{y \neq 0} \frac{\|\Sigma y\|}{\|y\|}$$

$$\Sigma y = \begin{pmatrix} \sigma_1 y_1 \\ \vdots \\ \sigma_n y_n \end{pmatrix} \Rightarrow \|\Sigma y\|^2 = \sum (\sigma_i y_i)^2$$

by $\sigma_1 \geq \sigma_2 \geq \dots \sigma_n \geq 0$

$$\Rightarrow \|\Sigma y\|^2 \leq \sigma_1^2 \sum_{i=1}^{n} y_i^2 = \sigma_1^2 \|y\|^2$$

$\|A\|_2 = \max_{y \neq 0} \frac{\|\Sigma y\|}{\|y\|} \leq \sigma_1$. Equality if $\exists y$ s.t. $\|\Sigma y\| = \sigma_1$

$\Rightarrow$ holds if $y = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\Rightarrow$ $\|A\|_2 = \sigma_1$

<u>On condition numbers</u>: $\kappa(A) = \|A\| \|A^{-1}\|$ for A square

$$\|A\| = \sigma_1$$

$$A^{-1} = (U \Sigma V^*)^{-1} = V^{-*} \Sigma^{-1} U^{-1} = \underbrace{V \Sigma^{-1} U^*}$$

another SVD for A^{-1} with singular values $= \frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_n} > 0$

largest.

$$\|A^{-1}\| = \frac{1}{\sigma_n}$$

$$\kappa(A) = \|A\| \|A^{-1}\| = \boxed{\frac{\sigma_1}{\sigma_n}}$$ also works for rectangular matrices!

($\frac{\sigma_1}{\sigma_r}$ for singular matrices "kind of" behaves like a condition number...)

<u>SVD & FTLA</u>: $A \in \mathbb{C}^{m \times n}$, $m \geq n$

$$A = \underbrace{U\Sigma}\underbrace{V^*} \Rightarrow \begin{bmatrix} | & & | \\ \sigma_1 u_1 & \cdots & \sigma_n u_n \\ | & & | \end{bmatrix} \begin{bmatrix} — v_1^* — \\ \vdots \\ — v_n^* — \end{bmatrix}$$

$$Ax = U\Sigma(V^* x) = U\Sigma \begin{bmatrix} v_1^* x \\ \vdots \\ v_n^* x \end{bmatrix} = \underbrace{\sum_{j=1}^{n} \sigma_j u_j v_j^* x}_{\text{dyadic form}}$$

$$Ax = \sum_{j=1}^{n} \underbrace{\sigma_j}_{\text{scalar}} u_j \underbrace{(v_j^* x)}_{\text{scalar}} \Rightarrow$$ Ax = linear combo of $u_1, \ldots, u_r$

$\sigma_1, \ldots, \sigma_r \neq 0$

$r = \text{rank}(A)$ (or number of nonzero σ_j).

$$R(A) = \{Ax : x \in \mathbb{C}^n\} = \text{span}\{u_1, \ldots, u_r\}$$

$u_1, \ldots, u_r$ = orthonormal basis for $R(A)$.

$$R(A) \perp N(A^*), \quad R(A) \oplus N(A^*) = \mathbb{C}^m$$

$$\dim(N(A^*)) = m - \underbrace{\dim(R(A))}_{r}$$

$u_{r+1}, \ldots, u_m$ = $m - r$ vectors $\perp$ to $\text{span}\{u_1, \ldots, u_r\} = R(A)$

$\Rightarrow$ $\{u_{r+1}, \ldots, u_m\}$ orthonormal basis for $N(A^*)$

$\{v_1, \dots, v_r\}$ = orth. basis for $R(A^*)$
$\{v_{r+1}, \dots, v_n\}$ = orth. basis for $N(A)$

SVD reveals fundamental subspaces!

SVD for image compression
$\hookrightarrow$ SVD as revealing best low rank matrix approximation.

$$A = \sum_{i=1}^{k} \alpha_i x_i y_i^* \Rightarrow \text{rank } k \text{ approx.}$$

$$\downarrow$$

A has $m \times n$ entries

$O((m+n)k)$ entries.