$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^* \quad \text{since } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$$

$$= \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots \sigma_r u_r v_r^*$$

If $\boxed{A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^*} \approx A$ if $\sigma_{k+1} << 1$

<u>Thm</u>: $\min_{\text{rank}(X)=k} \| A - X \|_2 = \sigma_{k+1}$ & is achieved by $X = A_k$.

<u>Proof</u>: from Demmel sect. 3.2.3. Assume $k < r$

Let X = rank k, $X \in \mathbb{C}^{m \times n}$

By FTLA: $\mathbb{C}^n = \underbrace{R(X^*)}_{\dim = k} + \underbrace{N(X)}_{\dim\ n-k}$.

Now let $A = \sum_{i=1}^{r} \sigma_i u_i v_i^*$ $\{v_1, \dots, v_r\} \in \mathbb{C}^n$

Since $N(X) \subset \mathbb{C}^n$ & $\dim(N(X)) = n-k$ & $k < r$

$$\Rightarrow \underset{\dim\ n-k}{N(X)} \cap \text{span}\{\underbrace{v_1, \dots, v_{k+1}}_{\dim\ k+1}\} \neq \emptyset$$

Let $z \in N(X) \cap \text{span}\{v_1, \dots, v_{k+1}\}$

assume $\|z\| = 1$

$$\|A-X\| = \max_{\substack{\|x\|=1 \\ x\neq 0}} \|(A-X)x\| \geq \|(A-X)z\|$$

$$= \|Az - \cancel{Xz}\|$$

since $z \in N(X)$

$$= \|Az\|$$

$$= \left\| \sum_{i=1}^{k+1} \sigma_i u_i v_i^* z \right\| \quad \text{by } z \in \text{span}\{v_1, \dots, v_{k+1}\}$$

$$= \left\| \sum_{i=1}^{k+1} \sigma_i \gamma_i u_i \right\| \quad \text{w/ } \gamma_i = v_i^* z$$

Ignoring σ_i for now,

Note $\left\| \sum_{i=1}^{k+1} \gamma_i u_i \right\|^2 = \left(\sum_{i=1}^{k+1} \gamma_i u_i \right)^* \left(\sum \gamma_i u_i \right)$

$$= \sum_{i=1}^{k+1} \gamma_i^2 \quad \text{by } u_j^* u_k$$

Note that $\sum |\gamma_i|^2 = \sum_{i=1}^{k+1} (v_i^* z)^2 = \sum_{i=1}^{n} (v_i^* z)^2 = \left\| \begin{bmatrix} - v_1^* - \\ \vdots \\ - v_n^* - \end{bmatrix} [z] \right\|^2$

$$= \|V^* z\|^2 = \|z\| = 1$$

by V unitary

$$\left\| \sum_{i=1}^{k+1} \sigma_i \gamma_i u_i \right\|^2 = \sum_{i=1}^{k+1} \sigma_i^2 |\gamma_i|^2$$

$$\geq \sigma_{k+1}^2 \sum_{i=1}^{k+1} |\gamma_i|^2 = \sigma_{k+1}^2$$

$\Rightarrow \|A - X\| \geq \sigma_{k+1}$ for any rank k matrix X.

If $\exists X$ s.t. $\|A-X\| = \sigma_{k+1}$

$$\Rightarrow \min_{\mathrm{rank}(X)=k} \|A-X\| = \sigma_{k+1}$$

Note $X = A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^*$

$$\|A-A_k\| = \left\| \sum_{j=k+1}^{r} \sigma_j u_j v_j^* \right\|$$

$$= \sigma_{k+1}$$

Low rank matrix approx. to datasets.

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \\ | & | & & | \end{bmatrix} \approx \sum_{j=1}^{k} \sigma_j u_j v_j^* \quad (k \ll n)$$

Span $\{u_1, \ldots, u_r\} = R(A)$

$\Rightarrow$ span $\{u_1, \ldots, u_k\} \approx R(A)$ (very math-imprecise!)

$$\begin{array}{cc} & \text{Bill \#} \\ A = & \begin{bmatrix} | & | & & \\ | & | & & \\ | & | & & \\ | & | & & \\ | & | & & \end{bmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \text{ Members of} \\ \leftarrow \text{ Congress} \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \\ & \uparrow \ \uparrow \ \uparrow \ \uparrow \end{array}$$

span $\{v_1, \ldots, v_k\} \approx R(A^*)$

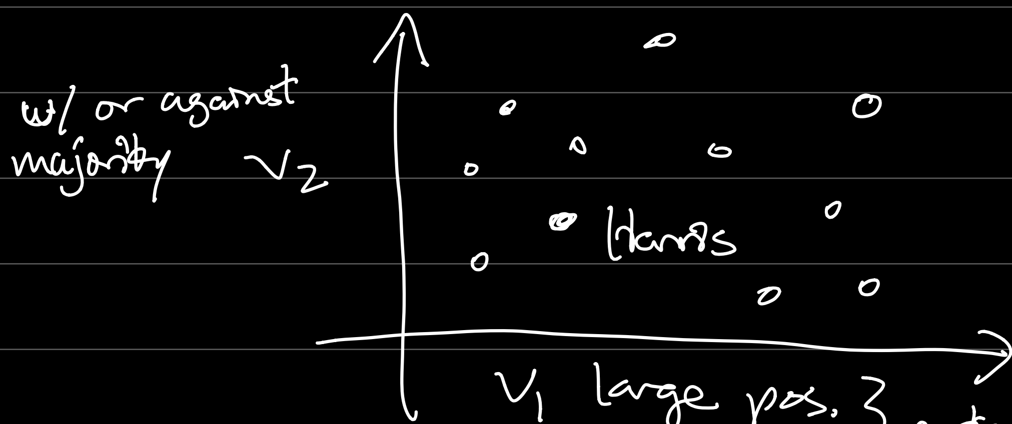
values of v_k corresponded to individual senators.

$\underbrace{u_1, u_2}$

template voting patterns

$\underbrace{v_1, v_2}$

tendencies for each senator to fall onto these 1st 2 template voting patterns.



v_1 large pos. } partisan
v_1 large neg. }

$v_1 = \begin{bmatrix} \\ \bullet \\ \\ \end{bmatrix}$ — Harris

$u_2 = \begin{bmatrix} \circ \end{bmatrix}$ — Harris