Today: Gaussian elimination as LU factorization

$\Rightarrow$ faster than QR, specific to square matrices.

**Gaussian elimination:** $Ax = b$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

QR factorization: $Q_{n-1} Q_{n-2} \cdots Q_1 A = R$

Treat LU similarly.

$$L_1 \overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{A} \longrightarrow$$ use row 1 to eliminate "c" in row 2

$$\Rightarrow \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix}$$ $\rightarrow$ mult. row 1 by $-\frac{c}{a}$, add to row 2.

$$L_1 = \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{bmatrix}}_{\text{unit lower triangular matrix}} . \quad L_1 A = \underbrace{\begin{bmatrix} a & b \\ 0 & d - b\frac{c}{a} \end{bmatrix}}_{U}$$

$$L_1 \underbrace{\begin{bmatrix} a_{ij} \rightarrow & \\ \downarrow & \searrow \end{bmatrix}}_{A} = L_1 \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & \\ \vdots & & & \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ 0 & \times & \times\times \\ 0 & \times & \times\times \\ 0 & \times & \times\times \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & & & \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ -\frac{a_{31}}{a_{11}} & & \ddots & \\ \vdots & & & \\ -\frac{a_{n1}}{a_{11}} & & & 1 \end{bmatrix}$$ unit lower triangular

Unit lower tri:

$$L_1 = \begin{bmatrix} 1 & \\ -\frac{c}{a} & 1 \end{bmatrix} \qquad L_1 L_1^{-1} = \begin{bmatrix} 1 & \\ -\frac{c}{a} & 1 \end{bmatrix} \overbrace{\begin{bmatrix} 1 & \\ \frac{c}{a} & 1 \end{bmatrix}}^{L_1^{-1}}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

works if larger than 2x2

$$L_1 = \underbrace{\begin{bmatrix} 1 & & \\ a & 1 & \\ b & & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 1 & & \\ -a & 1 & \\ -b & & 1 \end{bmatrix}}_{L_1^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_{n-1} L_{n-2} \cdots L_1 A = U$$

$$A = L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1} U$$

Product of $L_1^{-1} L_2^{-1}$ = lower triangular

$$\begin{bmatrix} 1 & & \\ a & 1 & \\ b & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

$$\Rightarrow A = \underbrace{L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1}}_{L} U = LU \text{ factorization}$$

$L$ = unit lower triangular

## 60s–80s implementations: Input $A$

output
$$\begin{bmatrix} u_{11} & u_{12} & - & - & - \\ l_{21} & u_{22} & - & - \\ l_{31} & l_{32} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & u_{nn} \end{bmatrix}$$

**Algorithm:** set $U = A$

for $k = 1, \dots, n-1$ (columns)
  for $j = k+1, \dots, n$ (rows below diag)
    $l_{j,k} = u_{j,k} / u_{k,k}$
    for $m = k, \dots, n$
      $u_{jm} = u_{jm} - l_{jk} u_{km}$

$A = LU$ (still uses backsub)

$Ax = b$

$$\Rightarrow \underbrace{L\underbrace{Ux}_{y} = b}\ \begin{cases} ① & Ly = b \\ ② & Ux = y \end{cases}$$

Advantages over QR: ① $QR \approx \frac{4}{3} n^3$ operations
② $LU \approx \frac{2}{3} n^3$ operations.

## Pitfalls of LU: numerical stability

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow$ LU divides by zero.

$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix}$ w/ $\frac{1}{\varepsilon} - 1 \approx \frac{1}{\varepsilon}$

$$A = LU = \begin{bmatrix} 1 & \\ 1/\varepsilon & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 1 \\ & 1 - 1/\varepsilon \end{bmatrix} = \begin{bmatrix} 1 & \\ 1/\varepsilon & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 1 \\ & -1/\varepsilon \end{bmatrix}$$

$$\underbrace{\begin{bmatrix}\varepsilon & 1\\ 1 & 1\end{bmatrix}}_{A} = LU \approx \begin{bmatrix}\varepsilon & 1\\ 1 & 0\end{bmatrix}.$$

Not conditioning since $\kappa(A) = O(1)$

$A = O(1)$

$$A^{-1} = \frac{1}{1-\varepsilon}\begin{bmatrix}-1 & 1\\ 1 & -\varepsilon\end{bmatrix} = O(1)$$

Solve issue by introduce <u>row swapping</u> $\Longrightarrow$ <u>pivoting</u>. Swap rows to ensure that the diagonal entry is the largest in below the diagonal $\Rightarrow$ $l_{jk} = u_{jk}/u_{kk} \leq 1$

$$A = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix} \qquad P_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$P_1 A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \Rightarrow LU. \quad P_1 A = LU$$

$\Rightarrow$ solve $Ax = b$

$$\underline{PA}x = Pb$$

$$LU\tilde{x} = Pb$$

$$x = P^{-1}x$$

For $A \in \mathbb{R}^{n\times n}$, $n > 2$, pivoting = messier

<u>Ex</u>: $n = 3$

$$A = \begin{bmatrix}1 & 2 & 4\\ 2 & 1 & -1\\ -1 & 4 & 2\end{bmatrix}.$$

$$P_1 = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad P_1 A = \begin{bmatrix}2 & 1 & -1\\ 1 & 2 & 4\\ -1 & 4 & 2\end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & & \\ -\tfrac{1}{2} & 1 & \\ \tfrac{1}{2} & 0 & 1 \end{bmatrix} \qquad L_1 P_1 A = \begin{bmatrix} 2 & 1 & -1 \\ & \tfrac{3}{2} & \tfrac{9}{2} \\ & \tfrac{9}{2} & \tfrac{3}{2} \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \qquad P_2 L_1 P_1 A = \begin{bmatrix} 2 & 1 & -1 \\ & \tfrac{9}{2} & \tfrac{3}{2} \\ & \tfrac{3}{2} & \tfrac{9}{2} \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & -\tfrac{1}{3} & 1 \end{bmatrix}, \quad L_2 P_2 L_1 P_1 A = \begin{bmatrix} 2 & 1 & -1 \\ & \tfrac{9}{2} & \tfrac{3}{2} \\ & & 4 \end{bmatrix} = U$$

This is partial pivoting = swap rows <u>below</u> diagonal

$$\Rightarrow \quad L_2 P_2 L_1 P_1 A = U$$

$$P_2 L_1 P_1 A = L_2^{-1} U$$

$$\uparrow$$

$$P_2^{-1} P_2$$

$$\Rightarrow \quad \underbrace{P_2 L_1 P_2^{-1}} P_2 P_1 A = L_2^{-1} U$$

$$L_1 = \begin{bmatrix} 1 & & \\ -\tfrac{1}{2} & 1 & \\ \tfrac{1}{2} & 0 & 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \rightarrow \text{elementary permutation matrix}$$

$$P_2 L_1 \underset{\overset{\shortparallel}{P_2}}{P_2^{-1}} = \begin{bmatrix} 1 & & \\ \tfrac{1}{2} & & 1 \\ -\tfrac{1}{2} & 1 & \end{bmatrix} \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$$

Define

$$P_2 L_1 P_2^{-1} = \begin{bmatrix} 1 & & \\ \tfrac{1}{2} & 1 & \\ -\tfrac{1}{2} & & 1 \end{bmatrix} = \tilde{L}_1$$

$$\underbrace{P_2 P_1}A = \underbrace{\tilde{L}_1^{-1} L_2^{-1}}U \Rightarrow \boxed{PA = LU}$$

Relies on fact that $P$ = elementary permutation

① $P^{-1} = P^T$

② $AP^{-1}$ swaps columns of $A$

③ $PAP^{-1}$ swaps both rows/cols of $A$.

If $A \in \mathbb{R}^{4\times4} \Rightarrow L_3 P_3 L_2 P_2 L_1 P_1 A = U$

$\tilde{L}_3 = L_3$

$$\Rightarrow \underbrace{P_3 L_2 P_3^{-1}}_{\tilde{L}_2} P_3 P_2 L_1 P_1 A = \tilde{L}_3^{-1} U$$

$$\Rightarrow P_3 P_2 L_1 P_1 A = \tilde{L}_2^{-1} \tilde{L}_3^{-1} U$$

$$\underbrace{P_3 P_2 L_1 P_2^{-1} P_3^{-1}}_{\tilde{L}_1} P_3 P_2 P_1 A = \tilde{L}_2^{-1} \tilde{L}_3^{-1} U$$

$\tilde{L}_1$ = unit lower tri.

$$\underbrace{P_3 P_2 P_1}A = \underbrace{\tilde{L}_1^{-1} \tilde{L}_2^{-1} \tilde{L}_3^{-1}} U$$

$$P \ A = LU$$