Last time: LU factorization w/ partial pivoting

$PA = LU \rightarrow$ needed properties of unit elementary lower triangular matrices.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$ fails w/out pivoting

How numerically stable is pivoting?

$A = LU$

$A + \delta A = \hat{L}\hat{U}$ with floating point roundoff

w/ $\frac{\|\delta A\|}{\|\hat{L}\|\,\|\hat{U}\|} = O(\varepsilon_{mach})$

$\|\delta A\| = O(\varepsilon_{mach})$ if $\|\hat{L}\|, \|\hat{U}\| = O(1)$

Note $|L_{ij}| \le 1$ b/c $l_{jk} = \frac{u_{jk}}{u_{kk}}$ & $u_{kk} \ge u_{jk}\ \forall j$ by pivoting.

$\Rightarrow \|\hat{L}\| = O(1)$

$$\Rightarrow \frac{\|\delta A\|}{\|A\|} \le \frac{\|\delta A\|}{\|A\|}\frac{\|\hat{U}\|}{\|\hat{U}\|}\frac{\overbrace{\|\hat{L}\|}^{\le C}}{\|\hat{L}\|} \le C \frac{\|\delta A\|}{\|\hat{U}\|\|\hat{L}\|}\frac{\|\hat{U}\|}{\|A\|}$$

$$\le C\, O(\varepsilon_{mach}) \frac{\|\hat{U}\|}{\|A\|}$$

Can est. $\frac{\|\hat{U}\|}{\|A\|} \le \text{Const} \times \rho$ w/ $\rho = \frac{\max |u_{jk}|}{\max |a_{jk}|}$

$$\frac{\|\delta A\|}{\|A\|} \le \rho\, O(\varepsilon_{mach}), \quad \rho = \text{amplification factor.}$$

Kahan matrix $\Rightarrow$ $\rho \le 2^{n-1}$ w/ $A \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & 1 & & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

add row 1 to row 2
add row 1 to row 3

1st column of LU $\Rightarrow$

$$\begin{bmatrix} 1 & & & & 1 \\ 0 & 1 & & & 2 \\ 0 & -1 & 1 & & 2 \\ 0 & -1 & -1 & 1 & 2 \\ 0 & -1 & -1 & -1 & 2 \end{bmatrix} \xrightarrow{\text{2nd column of LU}} \begin{bmatrix} 1 & & & & 2 \\ 0 & 1 & & & 2 \\ 0 & 0 & 1 & & 4 \\ 0 & 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & 1 \\ & 1 & & 2 \\ & & & 4 \\ & 0 & \ddots & \vdots \\ & & & 2^{n-1} \end{bmatrix} = U$$

QR, SVD exist for any matrix ($\mathbb{R}^{m\times n}$)
LU exists for any nonsingular square A
Cholesky or LDL^T " " " nonsingular square & symmetric pos (semi) definite

Suppose $A = A^*$, $x^* A x \geq 0 \quad \forall x \in \mathbb{C}^n$

$A = LU$ breaks symmetry in A!

$\hookrightarrow$ force L to be <u>unit</u> lower triangular

$$A \text{ SPD} \Rightarrow A = \begin{array}{c} \left[\begin{array}{c|c} \alpha & v^* \\ \hline v & C \end{array}\right] \begin{matrix} \\ n-1 \end{matrix} \\ \quad\quad n-1 \end{array}$$

One step of LU $\Rightarrow$

$$\underbrace{\begin{bmatrix} 1 & \\ -\frac{1}{\alpha} v & I \end{bmatrix}}_{\text{elem. unit lower tri. matrix.}} A = \left[\begin{array}{c|c} \alpha & v^* \\ \hline 0 & C - \frac{1}{\alpha} v v^* \end{array}\right]$$

$$A = \begin{bmatrix} 1 & \\ \frac{1}{\alpha} v & I \end{bmatrix} \left[\begin{array}{c|c} \alpha & v^* \\ \hline 0 & C - \frac{1}{\alpha} v v^* \end{array}\right]$$

$$= \begin{bmatrix} 1 & \\ \frac{1}{\alpha} v & I \end{bmatrix} \left[\begin{array}{c|c} 1 & \\ \hline & C - \frac{1}{\alpha} v v^* \end{array}\right] \left[\begin{array}{c|c} \alpha & v^* \\ \hline & I \end{array}\right]$$

move a factor of $\sqrt{\alpha}$ from right matrix to left matrix

$$= \underbrace{\left[\begin{array}{c|c} \sqrt{\alpha} & 0 \\ \hline \frac{1}{\sqrt{\alpha}} v & I \end{array}\right]}_{R_1} \underbrace{\left[\begin{array}{c|c} 1 & \\ \hline & C - \frac{1}{\alpha} v v^* \end{array}\right]}_{A_1} \underbrace{\left[\begin{array}{c|c} \sqrt{\alpha} & \frac{1}{\sqrt{\alpha}} v^* \\ \hline 0 & I \end{array}\right]}_{R_1^*}$$

Can "recurse" $\quad A = R_1 A_1 R_1^*$

$$= R_1 \left[\begin{array}{c|c} 1 & \\ \hline & C_1 \end{array}\right] R_1^* = R_1 \left[\begin{array}{c|c} 1 & \\ \hline & R_2 A_2 R_2^* \end{array}\right] R_1^*$$

$$A = \underbrace{R_1 \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{R}_2 \end{array}\right]}_{R_2} \left[\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & A_3 \end{array}\right] \underbrace{\left[\begin{array}{c|c} 1 & \\ \hline & \tilde{R}_2^* \end{array}\right]}_{R_2^*} R_1^*$$

$$= R_1 R_2 \cdots R_{n-1} R_{n-1}^* R_{n-2} \cdots R_1^*$$

$$= R R^* \quad , \quad R = \text{lower triangular}$$

Cholesky involves computing $C - \frac{1}{\alpha} vv^*$

where α = top left diagonal entry of the submatrix at the ith step of Cholesky.

$\Rightarrow$ Ensure $\alpha \neq 0$!

<u>1st step</u> $A = \left[\begin{array}{c|c} \alpha & v^* \\ \hline v & C \end{array}\right]$

<u>2nd step</u> $A = \cdots \left[\begin{array}{c|c} 1 & \\ \hline & C - \frac{1}{\alpha} vv^* \end{array}\right] \cdots$

$\hookrightarrow$ next α is 1st diagonal entry of this.

Can prove $\alpha \neq 0$ at any step if A = SPD.

<u>By induction</u> : $x^* A x > 0$

Base case) Let $x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $x^* A x = \alpha > 0$

Induction) Show $C - \frac{1}{\alpha} vv^*$ is still SPD

then Cholesky will never divide by zero.

Given $A = \left[\begin{array}{c|c} \alpha & v^* \\ \hline v & C \end{array}\right]$ = SPD, can prove $C - \frac{1}{\alpha} vv^* = $ SPD

$$u^*\left(C - \frac{1}{\alpha} vv^*\right)u > 0 \quad \forall u \in \mathbb{C}^{n-1}$$

But $x^*Ax > 0$: Let $x = \begin{bmatrix} \xi \\ y \end{bmatrix}$ $\quad \xi \in \mathbb{C}$, $y \in \mathbb{C}^{n-1}$

$$x^*Ax = \begin{bmatrix} \xi \\ y \end{bmatrix}^* \left[\begin{array}{c|c} \alpha & v^* \\ \hline v & C \end{array}\right] \begin{bmatrix} \xi \\ y \end{bmatrix} = \alpha \xi \bar{\xi} + \bar{\xi} v^* y + \xi y^* v + y^* C y > 0$$

Let $\xi = \dfrac{-v^* y}{\alpha}$

$$\Rightarrow \quad x^*Ax = \cancel{\alpha} \frac{(v^*y)^2}{\alpha^{\cancel{2}}} - \frac{(v^*y)^2}{\alpha} - \frac{(v^*y)^2}{\alpha} + y^*Cy$$

$$= -\frac{1}{\alpha}(v^*y)^2 + y^*Cy$$

$$= \underbrace{y^*Cy - \frac{1}{\alpha} y^* v v^* y} \quad > 0$$

$$= y^*\left(C - \frac{1}{\alpha} vv^*\right) y > 0 \quad \forall y \in \mathbb{C}^{n-1}$$

<u>Advantages</u>: cost & memory.

QR $\approx \frac{4}{3} n^3$

LU $\approx \frac{2}{3} n^3$

Cholesky $\approx \frac{1}{3} n^3$ w/ $\frac{1}{2} n(n+1)$ storage vs n^2 storage

Cholesky doesn't require pivoting & is fairly numerically stable to roundoff.

If $A = \hat{R}\hat{R}^*$ under floating pt.
& $\hat{R}\hat{R}^* x = b$, then x solves
$(A+E)x = b$ w/ $\frac{\|E\|}{\|A\|} \leq \varepsilon_m C_n$
Here, C_n = a constant depending on n.

Suli & Meyers prove C_n doesn't grow as fast as on LU. C_n is similar to ρ for LU
but $\underset{1 \leq k \leq i-1}{R_{ik}} < A_{ii}$ so C_n doesn't blow up like ρ does for LU.