Let $A \in \mathbb{C}^{n \times n}$ $\Rightarrow$

Eigenvalues + eigenvectors : $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^n$ s.t. $Av = \lambda v$

<u>Motivation</u>:
- Google PageRank
- graph clustering
- computational chemistry, vibration analysis, comp. phys. applications.

<u>Typical approach</u>: $\underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial in } \lambda} = 0$

<u>Ex</u>: $A = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$ $\quad \underbrace{(1-\lambda)(4-\lambda) - 6}_{\det(A)} = 0$

<u>Wilkinson</u>: computing eigenvalues via char. poly. = terrible.

$$A = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 20 \end{bmatrix} \qquad \lambda(A) = 1, 2, \dots, 20$$

$\hookrightarrow$ roundoff ($n = 21$, set imag. components)

Avoid computing λ, compute eigenvectors v.

$$Av = \lambda v \Rightarrow \text{recover } \lambda.$$

<u>Power iteration:</u> suppose A has eigenvalues
$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \cdots |\lambda_n| \geq 0$$

for x_0 = initial guess

Then $\underbrace{x_k = A^k}x_0 \Longrightarrow x_k \to v_1$

$$\begin{cases} x_1 = Ax_0, & x_1 = x_1/\|x_1\| \\ x_2 = Ax_1 & \\ \vdots & \end{cases} \Rightarrow \text{suppose } A = A^*$$

$$A = V \Lambda V^* \qquad V = \text{orthonormal matrix whose columns = eigenvectors}$$

$$= \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - v_1^* - \\ - v_2^* - \\ \vdots \\ - v_n^* - \end{bmatrix}$$

$$Ax = \lambda_1 (v_1^* x) v_1 + \lambda_2 (v_2^* x) v_2 + \cdots$$
$$A^2 x = \lambda_1^2 (v_1^* x) v_1 + \lambda_2^2 (v_2^* x) v_2 + \cdots$$
$$\vdots$$
$$\frac{A^k x}{\lambda_1^k} = \frac{\lambda_1^k c_1 v_1 + \lambda_2^k c_2 v_2 + \cdots}{\lambda_1^k}$$

$$= c_1 v_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k c_2 v_2 + \left(\frac{\lambda_3}{\lambda_1}\right)^k c_3 v_3 + \cdots$$

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \Rightarrow \left(\frac{|\lambda_2|}{|\lambda_1|}\right)^k \to 0 \quad \text{as } k \to \infty$$

$$\frac{A^k x}{\lambda_1^k} \to c_1 v_1 \quad \text{as } k \to \infty$$

If A has $\lambda_1, \ldots, \lambda_n$ eigenvalues.

$A - \mu I$ has $\lambda_1 - \mu, \lambda_2 - \mu, \ldots, \lambda_n - \mu$ eigenvalues.

A^{-1} has $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$ eigenvalues

Inverse iteration: $x_k = A^{-k} x_0 \begin{cases} x_1 = A^{-1} x_0 \\ x_2 = A^{-1} x_1 \\ \vdots \end{cases}$

as $k \to \infty$, $x_k \to c_n v_n$

Shifted inverse iteration: $x_k = (A - \mu I)^{-k} x_0$

Suppose μ is closest to λ_i

$x_k \to c_i v_i$

Rayleigh quotient iteration: shifted inverse iteration.

RQI

choose $\mu = \mu_k = \underbrace{\dfrac{x_{k-1}^* A x_{k-1}}{x_k^* x_k}}_{\text{Rayleigh quotient}}$

If $x_k = v_i$ $= i$th eigenvector $\Rightarrow \mu_k = \dfrac{v_i^* A v_i}{v_i^* v_i} = \lambda_i$

$(A - \mu_k I)^{-1}$ is more expensive to compute but RQI converges quadratically for general A.

If $A = A^*$, RQI converges cubically.

These iterations compute one eigenvalue at a time. Suppose iteration converges in 10 steps.

$\rightarrow$ $10n$ steps to compute full spectra

cost of one step $= O(n^2)$ for power iter

$= O(n^3)$ for RQI

$\Rightarrow$ $O(n^3)$ at best (really optimistic).

$O(n^4)$ at worst.

Strategy: transform A into form w/ easy to determine eigenvalues.

① diagonal matrix

$\rightarrow$ ② triangular matrix $\rightarrow$ λ_i = diagonal entries.

Transform A into a triangular matrix using similarity transforms.

Def: Let S invertible + square.

Then SAS^{-1} is a similarity transform of A.

SAS^{-1} has same eigs. as A

$\Rightarrow$ $SAS^{-1}w = \lambda w$

$A(S^{-1}w) = \lambda(S^{-1}w) \Rightarrow$ $\quad S^{-1}w = v$

$Av = \lambda v$

Unitary similarity transforms = very numerically stable.

Good news: Thm: Schur decomp.

$A \in \mathbb{C}^{n \times n}$, $\exists$ U unitary & upper tri T s.t. $A = UTU^*$

$(U^*AU = T) \Rightarrow$ reveals eigs of A = diag. entries of T.

Bad news:

What is hard about computing all eigenvalues at once? No explicit or analytical formula.

Recall eigenvalues = roots of a characteristic polynomial = degree n

Galois: Abel's impossibility thm: no algebraic solution for roots of any polynomial w/ degree > 4.

$\Rightarrow$ for $A \in \mathbb{C}^{n \times n}$ w/ $n > 4$

$\Rightarrow$ no closed form expression for the eigenvalues

$\Rightarrow$ no direct method for computing eigenvalues.

terminates in finite steps.

must be iterative & approx. eigs.