$A_0 = A$

for $i = 0, 1, 2, \ldots$

$$\left| \begin{array}{l} Q_k R_k = A_k \\ A_{k+1} = R_k Q_k \end{array} \right.$$

similarity transform SAS^{-1}

$$R_k = Q_k^* A_k$$

$$A_{k+1} = R_k Q_k = \underbrace{Q_k^* A_k Q_k}$$

preserve eigenvalues of A.

① Why does it work?

② How do you speed it up?

↳ reduce cost within each iter.

↳ accelerate QR iter for $O(1)$ iters.

QR iteration $\Leftrightarrow$ block power iteration.

single vector) Given x_0, $x_{k+1} = Ax_k$, $x_{k+1} = x_{k+1} / \|x_{k+1}\|$

Block) Given X_0. $X_{k+1} = AX_k$, $X_{k+1} = \text{orthonormalize}(X_{k+1})$

↳ QR, or other

$(X_{k+1}, R = qr(X_{k+1}))$

<u>Simultaneous iteration</u>

Let $\tilde{Q}_0 = I$

for $k = 1, 2 \ldots$

$$\left| \begin{array}{l} Z = A\tilde{Q}_{k-1} \\ \tilde{Q}_k R_k = Z \end{array} \right.$$

(normalization using QR)

$\Rightarrow$ produces $\tilde{Q}_k$.

<u>QR iteration</u>

Let $A_0 = A$

for $k = 0, 1, 2, \ldots$

$$\left.\begin{aligned} Q_{k+1} R_{k+1} &= A_k \\ A_{k+1} &= R_{k+1} Q_{k+1} \end{aligned}\right\} A_k$$

<u>Thm</u>: $\tilde{Q}_k = Q_1 Q_2 \cdots Q_k$

Define $\tilde{R}_k = R_k R_{k-1}, \ldots, R_1$

Then SI $\Leftrightarrow$ QR in the sense that

① $A_k = \tilde{Q}_k^* A \tilde{Q}_k$

② $\tilde{Q}_k \tilde{R}_k = A^k$

(<u>Proof</u> of ②) For SI, $\tilde{Q}_k R_k = A \tilde{Q}_{k-1}$

Define $R_0 = I$, $Q_0 = I$. $AQ_0 = A = \tilde{Q}_1 R_1$

$$AA = A\tilde{Q}_1 R_1 = \tilde{Q}_2 R_2 R_1$$

$$A^3 = A(AA) = \underbrace{A\tilde{Q}_2}\, R_2 R_1$$

$$= \tilde{Q}_3 R_3 R_2 R_1$$

$$A^4 = A(AAA) = A\tilde{Q}_3 R_3 R_2 R_1$$

$$= \tilde{Q}_4 R_4 R_3 R_2 R_1$$

$$A^k = \tilde{Q}_k \underbrace{R_k R_{k-1} \cdots R_1}_{\tilde{R}_k}$$

$$= \tilde{Q}_k \tilde{R}_k$$

(Proof of (i))

$$\underbrace{A_k}_{\text{from QR}} = \underbrace{\tilde{Q}_k}_{\text{from SI}}^* A \tilde{Q}_k$$

For QR $\Rightarrow$ $A = Q_1 R_1$

$$A^2 = Q_1 \underbrace{R_1 Q_1}_{} R_1$$

$$= Q_1 A_1 R_1$$

Recall $Q_2 R_2 = A_1$

$$\Rightarrow A^2 = \underbrace{Q_1 Q_2}_{\tilde{Q}_2} \underbrace{R_2 R_1}_{\tilde{R}_2}$$

$$A^k = \tilde{Q}_k \tilde{R}_k$$

w/ $\tilde{Q}_k = Q_1 Q_2, \ldots, Q_k$

QR, SI $\Longleftrightarrow$ $A^k = \tilde{Q}_k \tilde{R}_k$

can now show

$$A_k = \tilde{Q}_k^* A \tilde{Q}_k$$

for QR $\Rightarrow$ $A_k = R_k Q_k$

Since $Q_k R_k = A_{k-1} \Rightarrow R_k = Q_k^* A_{k-1}$

$$\Rightarrow A_k = Q_k^* A_{k-1} Q_k$$

$$A_k = Q_k^* Q_{k-1}^* A_{k-2} Q_{k-1} Q_k$$

$$= Q_k^* Q_{k-1}^* Q_{k-2}^* A_{k-3} Q_{k-2} Q_{k-1} Q_k$$

$$\vdots$$

$$A_k = \underbrace{Q_k^* Q_{k-1}^* \cdots Q_1^*} A \underbrace{Q_1 \cdots Q_k}$$

$$A_k = \tilde{Q}_k^* \quad A \quad \tilde{Q}_k$$

$\Rightarrow$ QR $\Leftrightarrow$ SI equiv. in that they both produce matrices related to the QR fac. of A^k.

<u>Speeding up QR:</u>

for $k = 1, 2, \ldots$

$Q_k R_k = A_{k-1}$ $\Rightarrow$ compute QR factorization $= O(n^3)$

$A_k = R_k Q_k$

Idea: pre-factorize $A \Rightarrow$ Upper Hessenberg