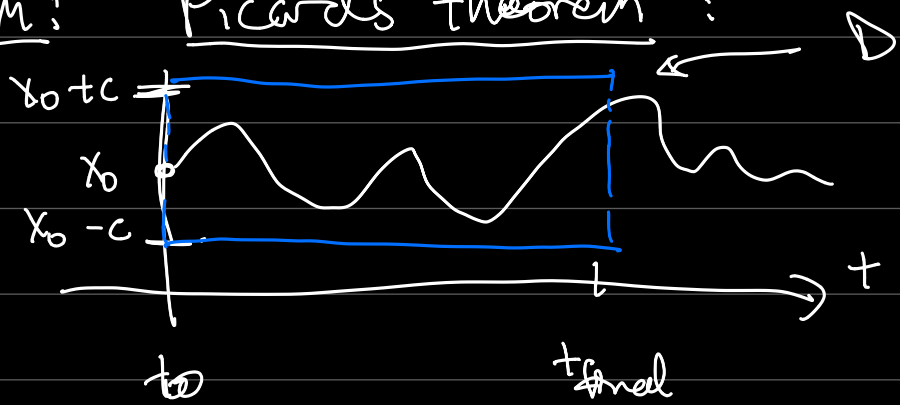
One-step methods: forward Euler, Improved Euler / Heun's method, backwards Euler, trapezoidal rule

$$x_{k+1} = x_k + h\,\varphi(x_k, t_k, h)$$

$$T_k = \frac{x(t_{k+1}) - x(t_k)}{h} - \varphi(x(t_k), t_k, h)$$

Thm: **Picard's theorem:**



$$\frac{dx(t)}{dt} = f(x,t)$$

1. Let $f(x,t)$ be continuous on $D = [x_0 - c, x_0 + c] \times [t_0, t_{final}]$ for c which is fixed.
2. Let $|f(x,t)| \le k$ for $x, t \in D$
3. Assume Lipschitz continuity $\Rightarrow \exists\, L > 0$ s.t. $|f(u,t) - f(v,t)| \le L|u - v|$ $\forall\, u, v \in [x_0 - c, x_0 + c]$ $\forall\, t \in [t_0, t_{final}]$
4. Assume $c \ge \frac{k}{L}\left(e^{L(t_{final} - t_0)} - 1\right)$

depends on k, L

$\Rightarrow \exists !\ x \in C^1[t_0, t_{final}]$ s.t.

$x(0) = x_0$ and $\frac{dx}{dt} = f(x,t)$ on $[t_0, t_{final}]$

Furthermore, $|x(t) - x(0)| \leq C$
$\forall\, t \in [t_0, t_{final}]$

<u>Global error analysis:</u> Assume $T_k \to 0$
as $h \to 0$ (necessary condition for convergence)

<u>Def:</u> a consistent method has $T_k \to 0$ as $h \to 0$

$\hookrightarrow$ for any ε, $\exists\, h(\varepsilon)$ s.t. $|T_k| < \varepsilon$
for all $0 < h < h(\varepsilon)$ and for
all $(x(t_k), t_k)$ and $(x(t_{k+1}), t_{k+1})$ in
$[x_0 - C, x_0 + C] \times [t_0, t_{final}]$

$$x_{k+1} = x_k + h\varphi(x_k, t_k, h)$$

since $T_k = \frac{x(t_{k+1}) - x(t_k)}{h} - \varphi(x(t_k), t_k, h)$

$$x(t_{k+1}) = x(t_k) + h\varphi(x(t_k), t_k, h) + hT_k$$

$$\underbrace{x(t_{k+1}) - x_{k+1}}_{e_{k+1}} = \underbrace{x(t_k) - x_k}_{e_k} + h\big(\varphi(x(t_k), t_k, h) - \varphi(x_k, t_k, h) + T_k\big)$$

Picard's: Lipschitz continuity on $f(x,t)$
$\Rightarrow$ assume similar conditions on φ.

$$|\varphi(u,t,h) - \varphi(v,t,h)| \leq L_\varphi |u-v|$$

$$\begin{aligned}
|e_{k+1}| &\leq |e_k| + h\left(|\varphi(x(t_k),t_k,h) - \varphi(x_k,t_k,h)| + |T_k|\right) \\
&\leq |e_k| + h\left(L_\varphi |x(t_k) - x_k| + |T_k|\right) \\
&= |e_k| + h L_\varphi |e_k| + |T_k| \\
&= (1 + hL_\varphi)|e_k| + |T_k|
\end{aligned}$$

max trunc. error

Picard's: $|f(x,t)| \leq k \iff |T_k| \leq T$ for $0 \leq k \leq n$

$$|e_{k+1}| \leq (1 + hL_\varphi)|e_k| + T$$

$$\begin{aligned}
|e_0| &= 0 \qquad (x(0) = x_0) \\
|e_1| &= h|T_0| \leq hT \\
|e_2| &= (1+hL_\varphi)|e_1| + hT \\
&\leq (1+hL_\varphi)hT + hT \\
|e_3| &\leq (1+hL_\varphi)|e_2| + hT \\
&\leq (1+hL_\varphi)\left((1+hL_\varphi)hT + hT\right) + hT \\
&\leq (1+hL_\varphi)^2 hT + (1+hL_\varphi)hT + hT
\end{aligned}$$

$$|e_n| \le hT \sum_{k=0}^{n-1} (1+hL_e)^k \Rightarrow \text{geometric series}$$

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$$

$$= hT \frac{|1-(1+hL_e)^n|}{|1-(1+hL_e)|}$$

$$= \frac{T}{L_e} |1-(1+hL_e)^n|$$

$$e^{hL_e} = \sum_{k=0}^{\infty} \frac{(hL_e)^k}{k!} = 1 + hL_e + \underbrace{\sum_{k=1}^{\infty} \frac{(hL_e)^k}{k!}}_{\text{if } hL_e \ll 1 \text{ this} \approx 0}$$

also positive since $h>0$ ← if $hL_e \ll 1$ this ≈ 0

If $hL_e \ll 1$, $|e_n| \le \frac{T}{L_e} |e^{nhL_e} - 1|$

for equispaced timesteps, $nh = T_{final}$ $\le \frac{T}{L_e} |e^{T_{final} L_e} - 1|$

$$\Rightarrow |x(t_n) - x_n| \le \frac{T}{L_e} |e^{T_{final} L_e} - 1|$$

Note $e^{T_{final} L_e}$ bad, but in absolute sense.

Ex: $\frac{dx}{dt} = \lambda x \Rightarrow$ $\quad x(t_n) = x_0 e^{\lambda n h}$

$x(t) = x_0 e^{\lambda t}$