An m-step LMM : $\sum_{j=0}^{m} \alpha_j x_{k+j} = h \sum_{j=0}^{m} \beta_j f_{k+j}$

$\frac{dx}{dt} = f(x,t)$

Last time: LMM truncation error $T_k \Rightarrow$

$$\left(\sum_{j=0}^{m} \beta_j\right) T_k = \frac{\sum_{j=0}^{m} \alpha_j \overbrace{x(t_{k+j})}^{x(t_k+jh)} - h\beta_j \overbrace{f(x_{k+j}, t_{k+j})}^{x'(t_k+jh)}}{h}$$

need $=0$ for $T_k \not\Rightarrow \infty$ as $h \to 0$

$$(\Sigma\beta_j)T_k = \frac{1}{h}\boxed{\sum_{j=0}^{m} \alpha_j}\, x(t_k)$$
$$+ \sum_{l=0}^{\infty} h^l \left(\sum_{j=0}^{m} \frac{\alpha_j j^{l+1}}{(l+1)!} - \beta_j \frac{j^l}{l!} \right) x^{(l+1)}(t_k)$$

$$= \frac{1}{h}\boxed{\sum_{j=0}^{m} \alpha_j}\, x(t_k) \longrightarrow \text{Need } \boxed{\sum_{j=0}^{m} \alpha_j = 0}$$

$$+ \boxed{\sum_{j=0}^{m} (j\alpha_j - \beta_j)}\, x'(t_k) \longrightarrow \boxed{\sum_{j=0}^{m} (j\alpha_j - \beta_j) = 0}$$

Consistency condition

$$+ h\boxed{\sum_{j=0}^{m} \left(\frac{j^2}{2}\alpha_j - j\beta_j \right)}\, x''(t_k)$$

$$+ h^2\boxed{\sum_{j=0}^{m} \left(\frac{j^3}{6}\alpha_j - \frac{j^2}{2}\beta_j \right)}\, x'''(t_k)$$

$$+ h^3\boxed{\sum_{j=0}^{m} \left(\frac{j^4}{24}\alpha_j - \frac{j^3}{6}\beta_j \right)}\, x''''(t_k) + \ldots$$

Def: a consistent LMM has $T_k \to 0$ as $h \to 0$

Thm: A LMM is consistent iff

$$\sum_{j=0}^{m} \alpha_j = \sum_{j=0}^{m} j\alpha_j - \beta_j = 0$$

Def: A LMM is has order of accuracy p if $T_k = O(h^p)$ as $h \to 0$

Thm: An m-step LMM is order $p>1$ iff
1. it is consistent
2. $$\sum_{j=0}^{m} \left(\frac{j^{l+1}}{(l+1)!} \alpha_j - \frac{j^l}{l!} \beta_j \right) = 0$$

for $l = 1, 2, \ldots, p-1$.

Ex: $\begin{pmatrix}\text{Fwd.}\\ \text{Euler}\end{pmatrix}$ $x_{k+1} - x_k = h f_k$ $\quad \alpha_0 = -1 \quad \beta_0 = 1$, $\quad \alpha_1 = 1 \quad \beta_1 = 0$

consistency $\begin{cases} \sum \alpha_j = 0 \\ \sum (j\alpha_j - \beta_j) = \alpha_1 + \alpha_0 \cdot 0 - \beta_1 - \beta_0 \\ \quad = 1 - 1 = 0 \end{cases}$

Is it 2nd order? $\sum \frac{j^2}{2} \alpha_j - j\beta_j = \frac{1}{2}\alpha_1 - \beta_1 = \frac{1}{2} - 1 = -\frac{1}{2}$

Ex: Trapezoidal rule

$$x_{k+1} - x_k = \frac{h}{2}(f_k + f_{k+1})$$

$$\alpha_0 = -1 \qquad \beta_0 = \frac{1}{2}$$
$$\alpha_1 = 1 \qquad \beta_1 = \frac{1}{2}$$

consistency
$$\begin{cases} \sum \alpha_j = 0 \\ \sum j\alpha_j - \beta_j = \alpha_1 + 0\alpha_0 - \beta_1 - \beta_2 \\ \qquad = 1 - \frac{1}{2} - \frac{1}{2} = 0 \end{cases}$$

$O(h^2)$ $\quad \sum \frac{j^2}{2}\alpha_j - j\beta_j = \frac{1}{2}\alpha_1 - 1\beta_1 = \frac{1}{2} - \frac{1}{2} = 0$

Not $O(h^3)$ $\quad \sum \frac{j^3}{6}\alpha_j - \frac{j^2}{2}\beta_j = \frac{1}{6}\alpha_1 - \frac{1}{2}\beta_1 = \frac{1}{6} - \frac{1}{2} \neq 0$

Stability:

1. $$x_{k+2} - \frac{3}{2}x_{k+1} + \frac{1}{2}x_k = \frac{h}{4}(5f_{k+1} - 3f_k)$$

$$\alpha_0 = \frac{1}{2} \qquad \beta_0 = -\frac{3}{4}$$
$$\alpha_1 = -\frac{3}{2} \qquad \beta_1 = \frac{5}{4}$$
$$\alpha_2 = 1 \qquad \beta_2 = 0$$

2. $$x_{k+2} - 3x_{k+1} + 2x_k = \frac{h}{2}(f_{k+1} - 3f_k)$$

$$\alpha_0 = 2 \qquad \beta_0 = -\frac{3}{2}$$
$$\alpha_1 = -3 \qquad \beta_1 = \frac{1}{2}$$
$$\alpha_2 = 1 \qquad \beta_2 = 0$$

$\Rightarrow$ Both are consistent & 2nd order accurate.

Ex: $x'(t) = x(t) \implies x(t) = x(0)e^{t}$

Method 1 works, method 2 blows up

$x'(t) = 0 \implies x(t) = x(0)$

Method 1 works, method 2 blows up

use this to analyze an LMM

$f(x,t) = 0 \implies f_k = 0$

$\implies$ LMM is just $\sum_{j=0}^{m} \alpha_j x_{k+j} = 0$

$\implies$ zero stable method doesn't admit $x_k \to \infty$ for $f_k = 0$