Last time: linear multistep methods

① $x_{k+2} - \frac{3}{2} x_{k+1} + \frac{1}{2} x_k = \frac{h}{4}(5f_{k+1} - 3f_k)$ looked stable

② $x_{k+2} - 3x_{k+1} + 2x_k = \frac{h}{2}(f_{k+1} - 3f_k)$ looked unstable

What makes these different?

Even blows up for $f(x,t) = 0$ if you perturb the initial iterates x_0, x_1 slightly (e.g. by 10^{-15}).

Assuming $f(x,t) = 0$

$$x_{k+2} - 3x_{k+1} + 2x_k = 0$$

$$x_{k+2} = 3x_{k+1} - 2x_k \Rightarrow \text{Analyze this.}$$

Similar update (constructed just for illustration)

$$x_{k+2} = 2x_k - x_{k+1}, \quad \text{assume } x_0 = 0, \; x_1 = \varepsilon$$

$$x_2 = 2(0) - \varepsilon = -\varepsilon$$

$$x_3 = 2x_1 - x_2 = 2\varepsilon + \varepsilon = 3\varepsilon$$

$x_4 = 2x_2 - x_3 = -2\varepsilon - 3\varepsilon = -5\varepsilon$

$x_5 = 2x_3 - x_4 = 6\varepsilon + 5\varepsilon = 11\varepsilon$

$x_6 \quad \ldots \quad = -21\varepsilon$

$x_7 \quad \ldots \quad = 43\varepsilon$

Sign alternates & roughly doubles in magnitude

$x_k \approx \left((-2)^k - 1\right)\varepsilon \Rightarrow$ take as ansatz

Assume $\underbrace{x_k = \gamma^k}_{\text{certain type of solution}}$ for γ constant

Plug into LMM : $x_{k+2} = 2x_k - x_{k+1}$

$\gamma^{k+2} = 2\gamma^k - \gamma^{k+1}$

$\Rightarrow$ if $\gamma \neq 0$ $\quad \gamma^2 = 2 - \gamma$

quad. eqn $\quad \gamma = -2$ or 1 (roots of quadratic)

Can have

$\Rightarrow \quad x_k = (-2)^k \Rightarrow$ $x_0 = 0$, $x_1 = \varepsilon$ admits a possible solution which blows up exponentially

Generalization: $\quad x_k = A\gamma_1^k + B\gamma_2^k$

$\gamma_1, \gamma_2 =$ const (we had 2 roots)

Plug into LMM.

$$\left(A\gamma_1^{k+2} + B\gamma_2^{k+2}\right) = 2\left(A\gamma_1^k + B\gamma_2^k\right) - \left(A\gamma_1^{k+1} + B\gamma_2^{k+1}\right)$$

for any initial history x_0, x_1, can derive A, B, γ_1, γ_2 are.

$$x_0 = 0, \quad x_1 = \varepsilon \Rightarrow \gamma_1, \gamma_2 = -2, 1$$

$$x_k = A\gamma_1^k + B\gamma_2^k$$

$$\Rightarrow \quad x_0 = A\gamma_1^0 + B\gamma_2^0 = A + B = 0$$

$$x_1 = A\gamma_1 + B\gamma_2 = \varepsilon$$

$$\Rightarrow \quad A = -\varepsilon/3, \quad B = \varepsilon/3$$

$$x_k = \frac{\varepsilon}{3} - \frac{\varepsilon}{3}(-2)^k$$

Big picture: looked for expressions for x_k which admit exponential growth

$\Rightarrow$ Plug into LMM update $\Rightarrow$ constants of exponential growth are related to roots of a polynomial equation.

$\Rightarrow$ for $x_{k+2} = 2x_k - x_{k+1} \Rightarrow \gamma = -2, 1$

$\Rightarrow \gamma = -2 \Rightarrow x_k \approx (-2)^k \to \infty$

$$\begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$$ Jose's approach

the matrix A has eigenvalues $-2, 1$

$$\det(A - \lambda I) = 0 \quad = \quad \text{characteristic polynomial}$$

$$\lambda^2 + \lambda - 2 = 0$$

<u>General LMM</u>:

$$\sum \alpha_j x_{k+j} = \underbrace{h \sum \beta_j f_{k+j}}_{=0} \quad \text{for } f(x,t) = 0$$

$$\sum_{j=0}^{m} \alpha_j x_{k+j} = 0$$

$$\Rightarrow \quad x_{k+m} = \frac{-1}{\alpha_m} \sum_{j=0}^{m-1} \alpha_j x_{k+j}$$

$$\begin{bmatrix} x_{k+1} \\ x_{k+2} \\ \\ x_{k+m-1} \\ x_{k+m} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ \underbrace{-\frac{\alpha_0}{\alpha_m}}_{c_0} & \underbrace{-\frac{\alpha_1}{\alpha_m}}_{c_1} & \cdots & \cdots & \underbrace{-\frac{\alpha_{m-1}}{\alpha_m}}_{c_{m-1}} \end{bmatrix}}_{A} \begin{bmatrix} x_k \\ x_{k+1} \\ \vdots \\ \vdots \\ x_{k+m-1} \end{bmatrix}$$

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots c_{m-1} z^{m-1} + z^m$$

Roots of $p(z)$ = eigenvalues of A.
(companion matrix)

Multiply $p(z)$ by $-\alpha_m \Rightarrow \rho(z)$

Def: $$\boxed{\rho(z) = \sum_{j=0}^{m} \alpha_j z^j}$$ Characteristic polynomial of a LMM.

Thm: A LMM is zero stable if roots γ_j of $\rho(z)$ satisfy

① $|\gamma_j| \le 1$ for all $j = 1, \dots, m$
② If $|\gamma_j| = 1$, the root is simple.

$\Rightarrow$ necessary & sufficient for zero stability.

Dahlquist Equivalence Thm: Suppose an m-step LMM is applied to a "well-behaved" IVP on $[t_0, t_{final}]$ w/ consistent starting values

$x_k \to x(t_k)$ for $k = 0, \dots, m-1$

$t_k = t_0 + hk$

then the method is convergent as $h \to 0$ iff the LMM is consistent and zero stable.

Absolute stability: zero stability misses β_j coefficients on the right hand side.

$\Rightarrow$ β_j imply a maximum h for stability & ODE