Last time: zero stability $\Rightarrow$ asymptotic stability as $h \to 0$. What about $h > 0$?

Absolute stability for LMMs

$\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \le 0$

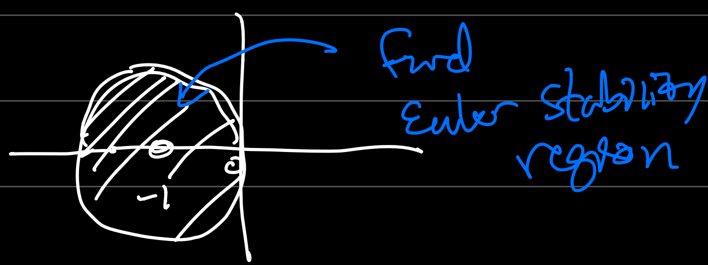
Ex: Fwd Euler applied to $x'(t) = \lambda x(t)$

$$x_{k+1} = x_k + h\lambda x_k = (1+h\lambda) x_k$$
$$= (1+h\lambda)^{k+1} x_0$$

for $x_{k+1} \not\to \infty$ as $k \to \infty$

need $|1+h\lambda| \le 1$

Need $h\lambda$ to be in a "region of stability"



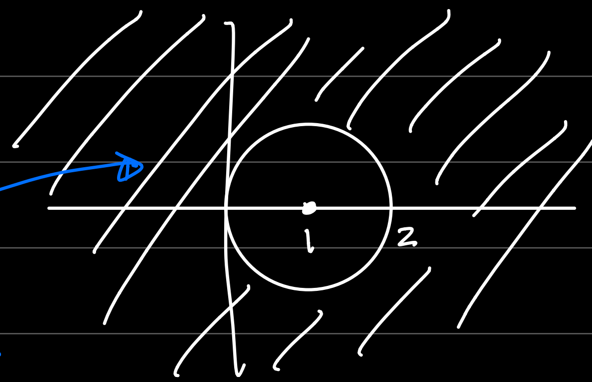
Ex: Backwards Euler, $x'(t) = \lambda x(t)$

$$x_{k+1} = x_k + h f_{k+1} = x_k + h\lambda x_{k+1}$$

$$(1-h\lambda)\, x_{k+1} = x_k \longrightarrow x_{k+1} = (1-h\lambda)^{-(k+1)} x_0$$

Need $\dfrac{1}{|1-h\lambda|} \le 1$ for $x_k \not\to \infty$

$$|1 - h\lambda| \geq 1$$



stability region for Backward Euler

Ex: trap. rule
$$x_{k+1} = x_k + \frac{h}{2}(f_k + f_{k+1})$$
$$= x_k + \frac{h}{2}(\lambda x_k + \lambda x_{k+1})$$
$$x_{k+1} = \frac{\left(1 + \frac{h\lambda}{2}\right)}{\left(1 - \frac{h\lambda}{2}\right)} x_k \quad ???$$

Reuse zero stability ideas.

For a general LMM: $\sum_{j=0}^{m} \alpha_j x_{k+j} = h \sum_{j=0}^{m} \beta_j f_{k+j}$

Assume $f_k = \lambda x_k$ $\quad (x'(t) = \lambda x(t))$

$$\sum_{j=0}^{m} \alpha_j x_{k+j} - h\beta_j \lambda x_{k+j} = 0$$

$$\Rightarrow \sum_{j=0}^{m} (\alpha_j - \beta_j h\lambda) x_{k+j} = 0$$

For zero stab., $\sum_{j=0}^{m} \alpha_j x_{k+j} = 0$ is stable if the roots γ_j of the characteristic polynomial $\rho(z) = \sum_{j=0}^{m} \alpha_j z^j$ satisfied $|\gamma_j| \leq 1$, & if $|\gamma_j| = 1$ then γ_j = simple.

By extension, LMM is absolutely stable if roots γ_j of $\sum_{j=0}^{m} (\alpha_j - \beta_j h\lambda) z^j$

satisfy $|\gamma_j| \leq 1$, & if $|\gamma_j| = 1$ then γ_j = simple.

Can avoid having to find roots for all $h\lambda$ by focusing instead on the <u>boundary</u> of the stability region.

$$\sum_{j=0}^{m} (\alpha_j - \beta_j h\lambda) z^j = \sum_{j=0}^{m} \alpha_j z^j - \underbrace{\left(\sum_{j=0}^{m} \beta_j z^j\right)}h\lambda$$

looking for roots of $\Rightarrow \rho(z) - h\lambda\, \sigma(z) = 0$

Roots: find z st. $\Rightarrow \rho(z) = h\lambda\, \sigma(z)$

Recall $\rightarrow$ we want roots $|\gamma_j| \leq 1$ (+ simple)

Roots satisfy $\dfrac{\rho(z)}{\sigma(z)} = h\lambda$. Look for $|z| = 1$

$\hookrightarrow$ determine the corresponding $h\lambda$.

<u>Ex</u>: Fwd Euler $\quad x_{k+1} - x_k = h f_k$

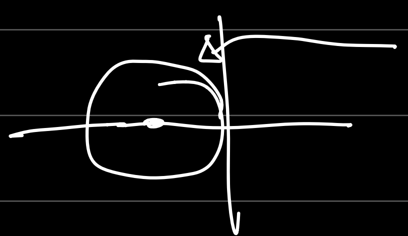
$\alpha_0 = -1 \qquad \beta_0 = 1$

$\alpha_1 = 1 \qquad \beta_1 = 0$

$$h\lambda = \frac{\rho(z)}{\sigma(z)} = \frac{z-1}{1} \Rightarrow \text{eval for } |z| = 1$$

(on boundary of the stab. region)

$$= z - 1$$



boundary of stab. region for fwd Euler is circle centered at -1.

To check inside/outside of stability region find roots $\rho(z) - h\lambda\,\sigma(z) = 0$ for a specific $h\lambda$ to test.

<u>Ex</u>: Trap. rule. $\quad x_{k+1} = x_k + \frac{h}{2}(f_k + f_{k+1})$

$$x_{k+1} - x_k = h\left(\frac{f_k}{2} + \frac{f_{k+1}}{2}\right)$$

$\alpha_0 = -1 \qquad \beta_0 = 1/2$

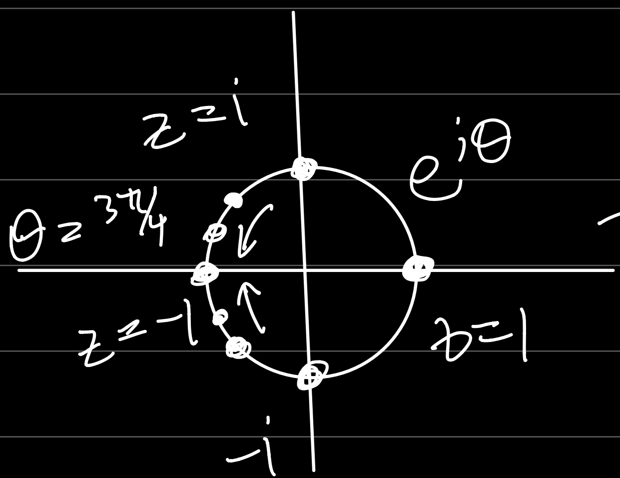
$\alpha_1 = 1 \qquad \beta_1 = 1/2$

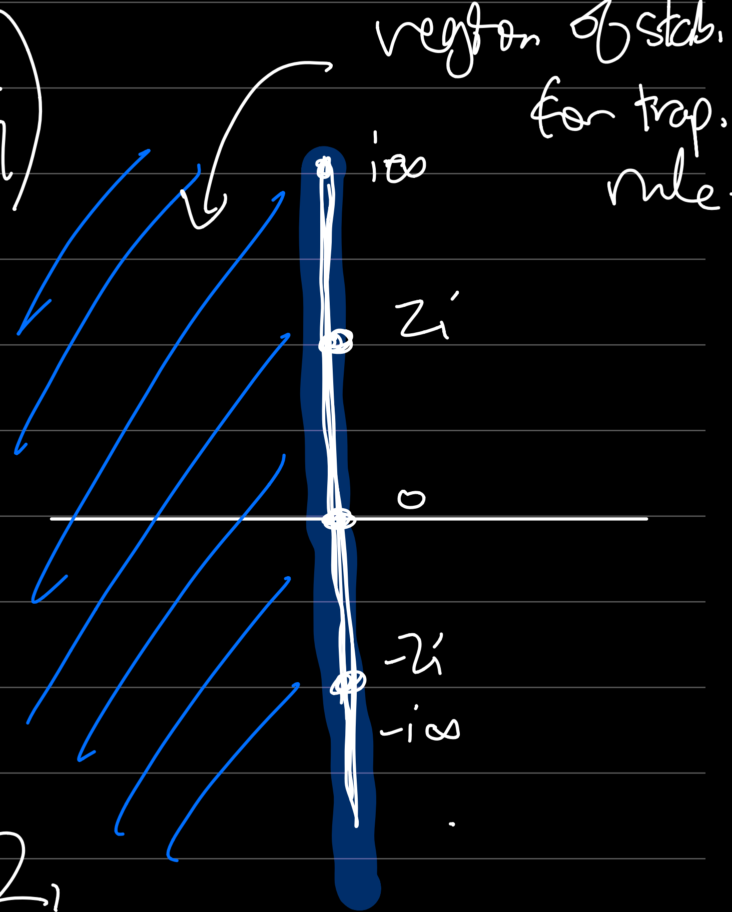
$$\rho(z) = z - 1$$

$$\sigma(z) = \sum \beta_j z^j = \frac{1}{2}(z+1)$$

Let $|z|=1$, $z=e^{i\theta}$ $\theta\in[0,2\pi)$

$$h\lambda = 2\left(\frac{z-1}{z+1}\right)$$





$$2\,\frac{i-1}{i+1} = 2i$$

Ex: AB-2

$$x_{k+2} - x_{k+1} = \frac{h}{2}\left(3f_{k+1} - f_k\right)$$

$\alpha_0 = 0$ $\beta_0 = -1/2$

$\alpha_1 = -1$ $\beta_1 = 3/2$

$\alpha_2 = 1$ $\beta_2 = 0$

stability boundary $h\lambda = \frac{\rho(z)}{\sigma(z)} = \frac{\sum \alpha_j z^j}{\sum \beta_j z^j} = \frac{z^2 - z}{\frac{3}{2}z - \frac{1}{2}}$

Useful fact: explicit method must have a finite stability region.

$\alpha_4 = -1$

$\alpha_5 = 1$

$\beta_0 = 251/720$

$\beta_1 = 1224/720$

$\beta_2 =$

$\beta_3 =$

$\beta_4 =$