Last time: stability regions for

$$x'(t) = \lambda x(t), \quad \lambda \in \mathbb{C}$$

$$\boxed{x(t) = e^{\lambda t} x(0) \to 0 \text{ as } t \to \infty} \longleftarrow \operatorname{Re}(\lambda) \le 0$$

Systems of ODEs: $x'(t) = Ax(t) \qquad x \in \mathbb{C}^n$

$x(0) = x_0 \qquad A \in \mathbb{C}^{n \times n}$

Assume A = diagonalizable.

$\Rightarrow A = V \Lambda V^{-1}$, V = matrix cols. are eigenvectors

Λ = diag. matrix of eigenvalues

$$x'(t) = \overbrace{V \Lambda V^{-1}}^{A} x(t)$$

$$\underbrace{V^{-1} x'(t)}_{} = \Lambda \underbrace{V^{-1} x(t)}_{}$$

$$y'(t) = \Lambda y(t) \qquad y(t) = V^{-1} x(t)$$

$$\left.\begin{aligned} y_1(t) &= \lambda_1 y_1(t) \\ y_2(t) &= \lambda_2 y_2(t) \\ &\vdots \\ y_n(t) &= \lambda_n y_n(t) \end{aligned}\right\} \quad y_j(t) = e^{\lambda_j t} y_j(0)$$

If we define $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$, $x(t) = V e^{\Lambda t} y(0)$

$$y(0) = V^{-1}x(0)$$

$$x(t) = \underbrace{V e^{\Lambda t}}\, V^{-1} x(0)$$

$$\boxed{x(t) = e^{At} x(0)} \longrightarrow 0 \quad \text{as } t \to \infty$$

if $Re(\lambda_j) \le 0$

<u>LMMs</u> : $x'(t) = \lambda x(t) \to$ needed
$\lambda h \in$ region of stability.

For $x'(t) = Ax(t)$, need $\lambda_j h \in$ region of stability
for all λ_j = eigenvalues of A.

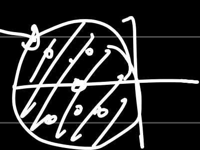
<u>Ex</u> :

$$\begin{aligned} x_{k+1} &= x_k + hAx_k \\ &= (I + hA)\, x_k \\ &= (I + hA)^{k+1} x_0 \end{aligned}$$

$$(I + hA)^{k+1} = V (I + h\Lambda)^{k+1} V^{-1}$$

$$\Rightarrow |1 + h\lambda_j| \le 1 \quad \forall j$$

$$\Rightarrow h\lambda_j$$



Stiff systems: If A has eigenvalues which are very different in magnitude.

Ex: $A = \begin{bmatrix} -1999 & -1998 \\ 999 & 998 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -100 & \\ & -1 \end{bmatrix}}_{\Omega} V^{-1}$

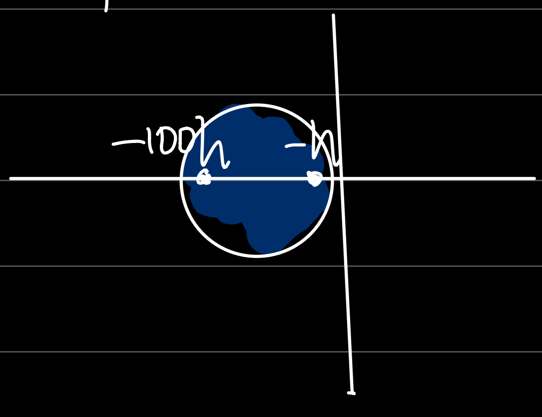
$$x(t) = e^{At}x(0) = V e^{\Lambda t} \underbrace{V^{-1}x(0)}_{y(0)} \quad y(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$x(t) = \underbrace{v_1 c_1 e^{-100t}}_{\to 0 \text{ very fast}} + \underbrace{v_2 c_2 e^{-t}}_{\text{decays slowly}}$$

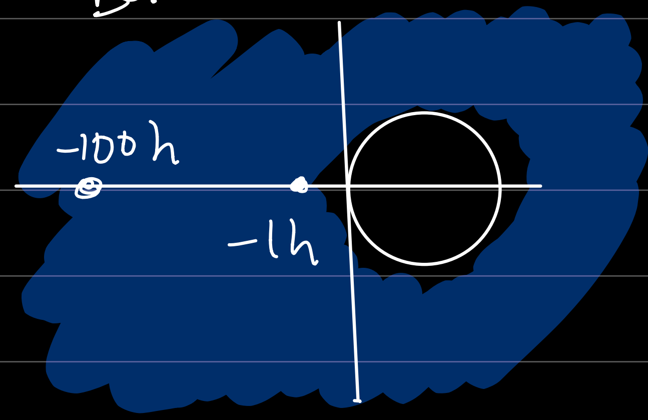
But still need $-100h$ lies inside region of stability.

For stiff systems, often explicit methods b/c they have finite regions of stability,

Fwd Euler



Backwards Euler



Def: a LMM is A-stable if stability region contains the entire left half plane.

$\Rightarrow$ If LMM is A-stable and $Re(\lambda_j) \leq 0 \Rightarrow$ LMM is stable for any $h > 0$.

Examples: backwards Euler
trap. rule is A-stable
implicit midpt. is A-stable

Bad news: ① only implicit methods can be A-stable

Dalquist's 2nd Barrier thm ② An A stable method is limited to 2nd order

Try to design implicit LMMs for stiff systems to be as stable as possible for $Re(\lambda_j)$ very negative.

$\hookrightarrow$ implicit Runge-Kutta or BDF

BDF = Backwards differentiation formulae.

↳ with $\beta_{k+j} = 0 \quad j \neq m.$

$$\sum_{j=1}^{m} \alpha_j x_{k+j} = \sum_{j=1}^{m} \beta_j f_{k+j} = \beta_m f_{k+m}$$

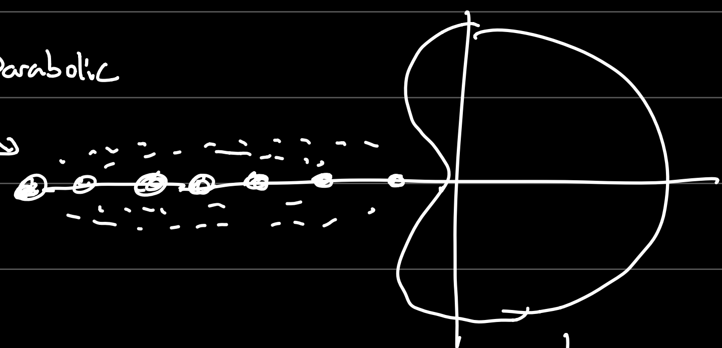
$\underline{m=1}$ $\quad x_{k+1} - x_k = h f_{k+1} \Rightarrow$ 1st order

$\underline{m=2}$ $\quad 3x_{k+2} - 4x_{k+1} + x_k = 2h f_{k+2} \Rightarrow$ 2nd order

$\underline{m=3}$ $\quad 11x_{k+3} - 18x_{k+2} + 9x_{k+1} - 2x_k = 6h f_{k+3} \Rightarrow$ 3rd order

For PDE's → parabolic

⇒ λ on negative real axis



Designed to behave like A-stable methods for certain PDEs.