What you should learn:

- analyze & understand the secant method.

Recall Newton: find root of $f(x)$ given x_0

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Secant iter:

$$x_{k+1} = x_k - \frac{f(x_k)(x_{k+1}-x_k)}{f(x_{k+1})-f(x_k)}$$

$$\left(f(x_k) \to 0 \qquad f'(x_k) \approx \frac{f(x_{k+1}) - f(x_k)}{x_{k+1}-x_k} \qquad \text{usual limit defn } h = x_{k+1}-x_k \right)$$

How do we analyze secant method?

1) secant converges (slowly)
2) prove a "rate" of convergence
3) infer a standard rate of convergence
   $\Rightarrow$ involves φ. (golden ratio).

Lemma: Let $\varepsilon \in (0,1)$ & let $f(\xi) = 0$ & $f'(\xi) \neq 0$, $f''(x)$ is continuous around ξ.

Then $\exists\, \delta > 0$ s.t. for any $a, b, c \in [\xi - \delta, \xi + \delta]$

$$\Rightarrow \left| 1 - \frac{(b-a)}{f(b)-f(a)} f'(c) \right| < \varepsilon$$

Pf. sketch: as $\delta \to 0$, $a \approx b \approx c \approx \xi$

$$f'(c) \to f'(\xi)$$

$$\frac{f(b)-f(a)}{b-a} \to f'(\xi)$$

Lemma: $\left| 1 - \frac{(b-a)}{f(b)-f(a)} f'(c) \right| < \varepsilon$

for $a, b, c \in [\xi - \delta, \xi + \delta]$

Thm (convergence): Let $\varepsilon \in (0,1)$ & $\delta > 0$ as in the prev. lemma.

Then, if $x_0, x_1 \in I_\delta = [\xi - \delta, \xi + \delta]$

$$\Rightarrow \quad x_k \to \xi \quad \& \quad \lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|} = 0$$

faster than linear convergence!

Error $e_k = x_k - \xi$

Pf:

Taylor's thm: $0 = f(\xi) = f(x_k) - f'(\eta_k)(x_k - \xi)$

$$0 = f(x_k) - f'(\eta_k)\, e_k$$

$$\Rightarrow \quad f(x_k) = f'(\eta_k)\, e_k$$

Secant:

$$\underbrace{(x_{k+1} - \xi)} = \underbrace{(x_k - \xi)} - f(x_k)\,\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

$$e_{k+1} = e_k - f(x_k)\,\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

$$e_{k+1} = e_k - f'(\eta_k)\,\frac{x_k - x_{k-1}}{\ldots}\, e_k$$

$x_{k+1} \quad x_k - f(x_k) \longrightarrow e_k$

$f(x_k) - f(x_{k-1})$

$$e_{k+1} = \underbrace{\left[1 - f'(\eta_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right]}_{\tilde{C}} e_k$$

like lemma $a, b, c = x_k, x_{k-1}, \eta_k$

by $\eta_k \in (x_k, \xi) \Rightarrow |\xi - \eta_k| < \delta$

$$|\tilde{C}| = \varepsilon < 1$$

$$|e_{k+1}| = |\tilde{C} e_k| \le |\tilde{C}|\,|e_k|$$

$$\le |\tilde{C}|^2 |e_{k-1}|$$

$$\le \underbrace{|\tilde{C}|^k}_{\to 0 \text{ as } k \to \infty} |e_0|$$

$$|e_{k+1}| \to 0 \Rightarrow x_{k+1} \to \xi$$

We have convergence!
What about rate?

$$\frac{e_{k+1}}{e_k} = \frac{e_k - f'(\eta_k)\dfrac{(x_k - x_{k-1})e_k}{f(x_k) - f(x_{k-1})}}{e_k}$$

$$\frac{|e_{k+1}|}{|e_k|} = \underbrace{\left| 1 - f'(\eta_k)\frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \right|}$$

$\rightarrow 0$ as $k \rightarrow \infty$

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k|} = 0$$

= faster than linear convergence!

Thm: Suppose $x_k \to \xi$ for secant

Then $$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = A = \left|\frac{f''(\xi)}{2f'(\xi)}\right|$$

Pf: $$\left(\begin{matrix} x_{k+1} \\ -\xi \end{matrix}\right) = \left(\begin{matrix} x_k \\ -\xi \end{matrix}\right) - f(x_k)\left(\frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})}\right)$$

$$e_{k+1} = e_k - f(x_k)\left(\frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})}\right)$$

$$e_{k+1} = \frac{e_k\left(f(x_k)-f(x_{k-1})\right) - f(x_k)(e_k - e_{k-1})}{\left(f(x_k)-f(x_{k-1})\right)}$$

$$\left(x_k - x_{k-1} = e_k - e_{k-1}\right)$$

$$e_{k+1} = \frac{e_{k-1}f(x_k) - e_k f(x_{k-1})}{f(x_k)-f(x_{k-1})}$$

$$= \underbrace{\frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})}} \; \frac{e_{k-1}f(x_k) - e_k f(x_{k-1})}{x_k - x_{k-1}}$$

$$= (\ast)\left(\frac{f(x_k)/e_k - f(x_{k-1})/e_{k-1}}{x_k - x_{k-1}}\right) e_k e_{k-1}$$

By Taylor: $f(x_k) = \cancel{f(\xi)}^{0} + f'(\xi)\overbrace{(\xi - x_k)}^{-e_k}$

$$+ \frac{f''(\eta_k)}{2}\underbrace{(\xi - x_k)^2}_{e_k^2}$$

$$\frac{f(x_k)}{e_k} = \frac{f''(\eta_k)}{2} e_k - f'(\xi)$$

$$\frac{f(x_{k-1})}{e_{k-1}} = \frac{f''(\eta_{k-1})}{2} e_{k-1} - f'(\xi)$$

Subbing into error eqn:

$$e_{k+1} = \underbrace{\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}}_{\boxed{\frac{1}{f'(\xi)}}}\left(\underbrace{\frac{\frac{f''(\eta_k)}{2}e_k - \frac{f''(\eta_{k-1})}{2}e_{k-1}}{x_k - x_{k-1}}}_{\frac{\boxed{\frac{f''(\xi)}{2}}(\cancel{e_k - e_{k-1}})}{\cancel{e_k - e_{k-1}}}}\right) e_k e_{k-1}$$

(since $f''(\eta_k) \to f''(\xi)$)

(by $x_k \to \xi$)

$$e_{k+1} = \frac{f''(\xi)}{2f'(\xi)} e_k e_{k-1}$$

$$\Longrightarrow \frac{e_{k+1}}{e_k e_{k-1}} = \frac{f''(\xi)}{2f'(\xi)}$$