What you should learn:

- how to derive secant convergence rate
- how to tell if and when a fixed point iteration converges (+ how fast)

Secant: $$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

Last time, we proved $\Rightarrow$ $$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = A = \left|\frac{f''(\xi)}{2f'(\xi)}\right| > 0$$

if $f'(\xi) \neq 0$ + f'' continuous near ξ

$\left(\frac{|e_{k+1}|}{|e_k||e_k|} = A \Rightarrow \text{quadratic convergence} \right)$

Thm: Assume secant converges (as above)

If $|e_{k+1}| = \gamma_k |e_k|^{\alpha}$ w/ $\gamma_k \to \gamma$

$\Rightarrow$ $\gamma = A^{1/\alpha}$ w/ α satisfying

$$\alpha^2 - \alpha - 1 = 0$$

$$\alpha = \frac{1+\sqrt{5}}{2} = \varphi, \text{ golden ratio.}$$

Pf: By $|e_k| = \gamma_{k-1} |e_{k-1}|^\alpha$

$$|e_{k-1}| = \left(\frac{|e_k|}{\gamma_{k-1}}\right)^{1/\alpha}$$

From secant convergence: $\underbrace{\frac{|e_{k+1}|}{|e_k||e_{k-1}|}} = \frac{|e_{k+1}|}{|e_k|\left(\frac{|e_k|}{\gamma_{k-1}}\right)^{1/\alpha}}$

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = A > 0$$

$$\Downarrow$$

$$\frac{\gamma_{k-1}^{1/\alpha}\gamma_k |e_k|^\alpha}{|e_k| \, |e_k|^{1/\alpha}}$$

$$\lim_{k\to\infty} \underbrace{\gamma_{k-1}^{1/\alpha}\gamma_k} |e_k|^{\alpha-1-1/\alpha} = A > 0$$

$$\left(\begin{array}{l} \text{by } \gamma_k \to \gamma \\ \Rightarrow \gamma^{1+1/\alpha} \end{array}\right)$$

since $|e_k| \to 0$

$$\alpha - 1 - \frac{1}{\alpha} = 0$$

$\Rightarrow$ mult by α $\quad \alpha^2 - \alpha - 1 = 0$

roots are $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow$ root $\alpha = \dfrac{1 \pm \sqrt{5}}{2}$

$\lim_{k\to\infty} |e_k|^{\alpha} \longrightarrow$ const $\Rightarrow$ α cannot be negative

$$\alpha = \frac{1+\sqrt{5}}{2}$$

$$\lim_{k\to\infty} \gamma_{k-1}^{1/\alpha} \gamma_k = A > 0$$

$$\lim_{k\to\infty} \frac{A}{\gamma_{k-1}^{1/\alpha} \gamma_k} = 1$$

By $\gamma_k \to \gamma$, $\gamma_{k-1}^{1/\alpha} \gamma_k = \gamma^{1+1/\alpha}$

$$A = \gamma^{1+1/\alpha}$$

$$\gamma = A^{1/(1+1/\alpha)} = \underbrace{A^{1/\alpha}}_{\text{true for } \alpha = \frac{1+\sqrt{5}}{2}}$$

Fixed point iteration

find α s.t. $\alpha = g(\alpha)$

$\Rightarrow \quad x_{k+1} = g(x_k)$

Stop when $|x_{k+1} - x_k| < tol$

Ex: Newton $x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{g(x_k)}$

$g(x) = x - \frac{f(x)}{f'(x)}$

$\Rightarrow \quad x_{k+1} = g(x_k)$

Fixed pt. iterations don't always converge!

Ex: ① $x = x^2 + x - 3$ (diverges)

$\Rightarrow x_k = 2, 3, 9, 86 \to \infty$

② $x = x - \frac{x^2 - 3}{2x} \Rightarrow x_k = 2, \frac{5}{4}, \approx 1.7,$

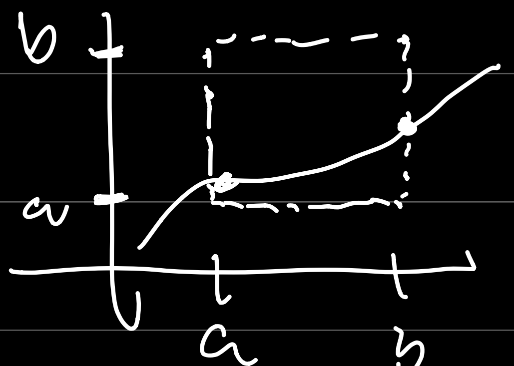
(converges)

③ $x = 3/x$, $x_k = 2, \frac{3}{2}, 2, \dots$
(oscillates)

<u>Lemma</u>: If $g \in C[a,b]$

& $g(x) \in [a,b]$ for $x \in [a,b]$

$\Rightarrow \exists \alpha \in [a,b]$ s.t. $\alpha = g(\alpha)$



<u>Pf</u>: Let $f(x) = x - g(x)$

$\alpha = g(\alpha) \Rightarrow f(\alpha) = 0$

$f(a) = a - \underbrace{g(a)}_{a \le (*) \le b} \le 0$

$f(b) = b - g(b) \ge 0$

$\Rightarrow f$ contin. $f(a)$ & $f(b)$ opposite signs

$\Rightarrow \exists$ root α inside $[a,b]$.

When does fixed point iter. converge?

$\Rightarrow$ when $g(x)$ is contraction mapping

<u>Def</u>: $g(x) \in \mathbb{R}$ is a contraction on $[a,b]$ if $\exists\ 0 < L < 1$

s.t. $|g(x) - g(y)| \leq L|x-y|$

for any $x, y \in [a,b]$,

$\Rightarrow$ Lipschitz contin. w/ $L \in (0,1)$

<u>Thm</u>: a) $g(x) \in C[a,b]$

b) $g(x) \in [a,b]$ for $\forall x \in [a,b]$

c) $g(x)$ is a contraction on $[a,b]$

$\Rightarrow$ unique fixed point $\alpha \in [a,b]$

& $x_{k+1} = g(x_k) \rightarrow \{x_k\} \rightarrow \alpha$

for any $x_0 \in [a,b]$

$\alpha = g(\alpha)$

Pf: By (a) & (b) $\Rightarrow$ fixed point exists.

Uniqueness $\rightarrow$ assume not unique

$\Rightarrow$ α, β both fixed points

$$|\alpha-\beta| = |g(\alpha) - g(\beta)|$$
$$\leq L|\alpha-\beta| < |\alpha-\beta|$$

$$\Rightarrow |\alpha-\beta| = 0 \Rightarrow \alpha=\beta.$$

For convergence: Let $x_0 \in [a,b]$.

$$|e_k| = |x_k-\alpha| = |g(x_{k-1})-g(\alpha)|$$
$$\leq L|x_{k-1}-\alpha|$$
$$= L|e_{k-1}|$$

$$|e_k| \leq L|e_{k-1}| \leq L^2|e_{k-2}| \leq \cdots$$
$$\leq \cdots L^k|e_0|$$

By $L<1$ $\Rightarrow$ $L^k \rightarrow 0$ as $k\rightarrow\infty$

$\Rightarrow$ $e_k \rightarrow 0$

Can replace contraction condition w/ condition on $g'(x)$

<u>Thm</u>: $\exists \alpha$ a fixed point of $g(x)$ in $[a,b]$.

Assume $g \in C^1(I_\delta)$, $I_\delta = [\alpha-\delta, \alpha+\delta]$ & $|g'(\alpha)| < 1$

$\Rightarrow$ the fixed point iteration converges to α if x_0 is "close enough".

<u>Pf</u>: Since g' is contin in I_δ & $|g'(\alpha)|<1$. $\exists h \leq \delta$ st.

$|g'(x)| \leq L < 1$ for $x \in I_h = [\alpha-h, \alpha+h]$

Let $x_k \in I_h$. Then, by M.V.T. over $[x_k, \alpha] \Rightarrow g'(\xi) = \dfrac{g(x_k)-g(\alpha)}{x_k-\alpha}$

$\exists \xi \in [x_k, \alpha]$

$$\underbrace{|x_{k+1}-\alpha|}_{|e_{k+1}|} = |g(x_k)-g(\alpha)|$$

$$\le |g'(\xi)|\,|x_k-\alpha|$$

$$\le L\,\underbrace{|x_k-\alpha|}_{|e_k|}, \quad L<1$$

$$|e_{k+1}| \le L\,|e_k| \rightarrow e_k \rightarrow 0$$