$x_{k+1} = \underbrace{x_k - (\cdots)}$? rootfinding

$x_{k+1} = g(x_k)$ fixed point iteration

Last time: conditions under which solution α exists & is unique $\Rightarrow$ if g is a contraction on some $[a,b] \subset \mathbb{R}$

$\rightarrow$ Fixed point iter. also provides a general way to look errors & convergence rate

Assume $g \in C^2([a,b])$

$$x_{k+1} = g(x_k) = \overbrace{g(\alpha)}^{\alpha} + g'(\alpha)\overbrace{(x_k - \alpha)}^{e_k} + \frac{g''(\eta_k)}{2}\underbrace{(x_k - \alpha)^2}_{e_k^2}$$

$(e_k = x_k - \alpha)$

$$e_{k+1} = x_{k+1} - \alpha = g'(\alpha)e_k + \frac{g''(\eta_k)}{2}e_k^2$$

(Note η_k = between x_k, α)

If $g'(\alpha) = 0$, $e_{k+1} = \frac{g''(\eta_k)}{2}e_k^2$

$\rightarrow$ can translate $g'(\alpha) = 0$ to Newton's method.

$$x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{g(x_k)}$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{(f'(x))^2}$$

$$g'(x) = \frac{f(x)\,f''(x)}{f'(x)^2}$$

If $g'(\alpha) = 0$ (note α = root for Newton)

$\Rightarrow$ $f'(\alpha) \neq 0$ (not a double root)

$f(\alpha) = 0$ (α = root)

$$\Rightarrow \lim_{k\to\infty} e_{k+1} = \frac{g''(\alpha)}{2} e_k^2$$

(quad. convergence)

Otherwise, if $f(\alpha) = 0$

but $f'(\alpha) = 0$

$$\lim_{x\to\alpha} g'(x) = \lim_{x\to\alpha} \frac{f(x)\,f''(x)}{f'(x)^2} \quad \text{(by L'Hopital)}$$

$$= \lim_{x\to\alpha} \frac{f'(x)\,f''(x) + f(x)\,f'''(x)}{2f'(x)\,f''(x)}$$

(by L'Hopital)

$$= \lim_{x\to\alpha} \frac{\left(f''(x)^2 + f'(x)\,f'''(x) + f'(x)\,f'''(x) + \cancel{f(x)}^{\,0} f^{(4)}(x)\right)}{2\left(f''(x)^2 + f'(x)\,f'''(x)\right)}$$

If $f''(x) \neq 0$ (not triple root)

$$= \lim_{x \to \alpha} \frac{f''(x)^2 + 2f'(x)f'''(x)}{2f''(x)^2 + f'(x)f'''(x)}$$

$$= \frac{f''(\alpha)^2}{2f''(\alpha)^2} = \frac{1}{2}$$

$$e_{k+1} = g'(\alpha)e_k + \frac{g''(\eta_k)}{2} e_k^2$$

$$\lim_{k \to \infty} \frac{e_{k+1}}{e_k} = \frac{1}{2} =$$ convergence rate for bisection.