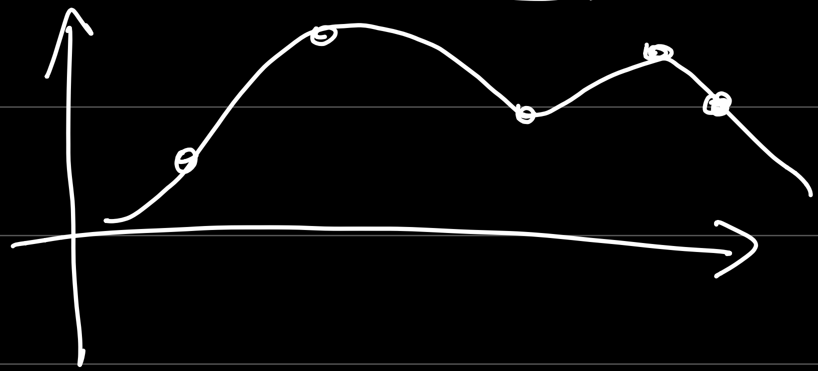
Polynomial interpolation



Goal: given $\{x_0, \dots, x_n\}$ pts

& $\{y_0, \dots, y_n\}$

$\Rightarrow$ find $p(x)$ s.t. $p(x_i) = y_i$ $i=0,\dots,n$

$p(x)$ polynomial.

Example w/ monomials:

degree n

$$p(x) = c_0 + c_1 x + c_2 x^2 + \dots c_n x^n$$

$= n+1$ coeffs.

$$p(x_i) = y_i \quad \text{for } i = 0, \dots, n$$

$$\Rightarrow c_0 + c_1 x_i + c_2 x_i^2 + \dots c_n x_i^n = y_i$$

($n+1$ coeffs, $n+1$ equations (linear)

Write as matrix eqn.

$$Vc = y \qquad V \in \mathbb{R}^{(n+1)\times(n+1)}$$
$$c, y \in \mathbb{R}^{n+1}$$

$$\begin{bmatrix} c_0 + c_1 x_0 + c_2 x_0^2 + \cdots \\ c_0 + c_1 x_1 + c_2 x_1^2 + \cdots \\ c_0 + c_1 x_2 + c_3 x_2^2 + \cdots \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & & x_2^n \\ \vdots & & & & \vdots \end{bmatrix}}_{\text{Vandermonde matrix}} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Solve $c = V^{-1} y$, can reconstruct

$$p(x) = \sum_{i=0}^{n} c_i x^n$$

given a list

$\Rightarrow$ of points $\bar{x} \in \mathbb{R}^p \Rightarrow V_p = \begin{bmatrix} 1 & \bar{x}_0 & \bar{x}_0^2 & \\ 1 & \bar{x}_1 & \bar{x}_1^2 & \cdots \\ 1 & \vdots & \vdots & \\ 1 & \bar{x}_p & \bar{x}_p^2 & \end{bmatrix}$

$V_p c \rightarrow$ values of $\begin{bmatrix} \vdots \\ p(\bar{x}_i) \\ \vdots \\ \vdots \end{bmatrix}$ $i = 1, \dots, p$

Prove that : ① interp. sol. is unique & exists

② interpolated polynomial $p(x) \rightarrow f(x)$ as $n \rightarrow \infty$

Def :