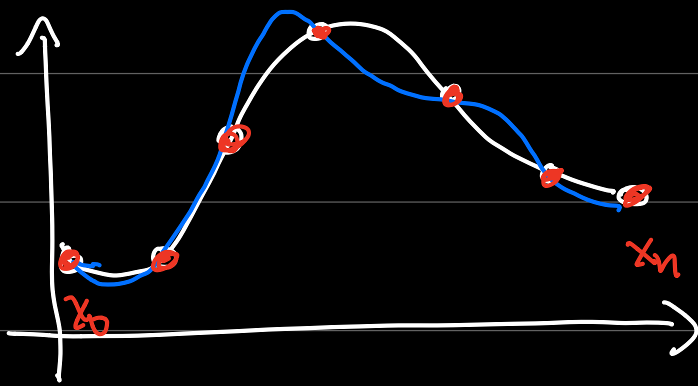
Last time: setting up polynomial interpolation



find $p_n(x)$ degree n

s.t. $p_n(x_i) = y_i$

for $i = 0, \dots, n$

- Rootfinding, fixed point iteration
- polynomial interpolation
  - ↳ how to set up & solve the problem
  - ↳ when the problem is solvable & when it's accurate
  - ↳ tailored techniques

→ Vandermonde system

$$p_n(x) = \sum_{i=0}^{n} c_i x^i$$

$$\Rightarrow \quad V \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

$V =$ Vandermonde matrix

$$V = \begin{bmatrix} x_0^0 & x_0^1 & \cdots & x_0^n \\ \vdots & \vdots & & \vdots \\ x_n^0 & x_n^1 & \cdots & x_n^n \end{bmatrix} \Longrightarrow \text{non-singular}$$

$\Downarrow$

rows are linearly indep.

points = distinct

<u>Thm</u>: for any set of distinct points $\{x_j\}_{j=0}^n$ $\{y_j\}_{j=0}^n$, $\exists!$ (exists a unique) polynomial $p_n \in P^n$ s.t. $p_n(x_i) = y_i$.

<u>Pf</u>: Assume $p_n(x), q_n(x)$ s.t. $p_n(x_i) = q_n(x_i) = y_i$, $i = 0, \ldots, n$

$$r_n(x) = p_n(x) - q_n(x) \in P^n$$

$$r_n(x_i) = 0, \quad i = 0, \ldots, n$$

$$\Rightarrow \quad r_n(x) = 0$$

<u>Thm</u>: Let $f \in C^{n+1}(I_t)$ for some $t \in \mathbb{R}$ where I_t = smallest interval containing $x_0, x_1, \ldots, x_n$ and t

$$\Rightarrow \quad I_t = H[x_0, \ldots, x_n, t]$$

$\Rightarrow \exists\, \alpha \in I_t$ s.t.

$$e(t) = f(t) - p_n(t)$$
$$= \frac{f^{(n+1)}(\alpha)}{(n+1)!}(t-x_0)(t-x_1)\cdots(t-x_n)$$

Pf: $t = x_j$, $e(t) = 0$

Assume $t \neq x_j$.

Define $\Psi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

& define $G(x) = e(x) - \frac{\Psi(x)}{\Psi(t)} e(t)$

Notice: 1) $G(x) \in C^{n+1}$

2) G has roots at $x_0, \dots, x_n$ & t.

$= n+2$ roots $\in I_t$

$\Rightarrow$ Mean value theorem.

MVT: $f \in C^1[a,b]$ & diff'ble on $(a,b) \Rightarrow \exists\, \eta \in (a,b)$ s.t.

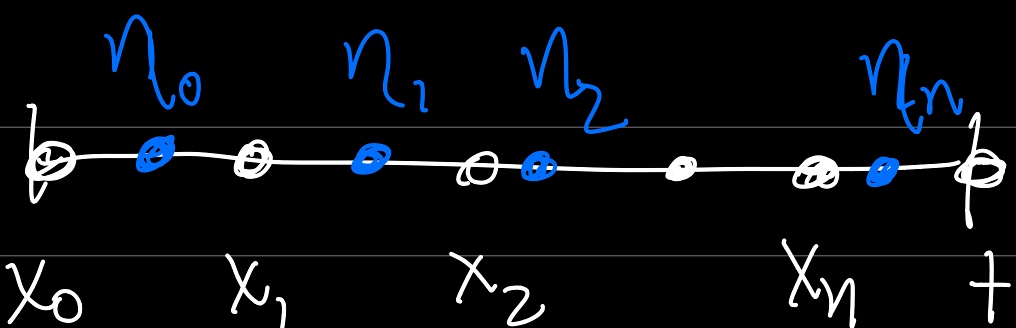
$$f'(\eta) = \frac{f(b)-f(a)}{b-a}$$

$\Rightarrow$ let a, b = any two pts from $\{x_0, \dots, x_n, t\}$

$\Rightarrow$ MVT for $G(x)$ over $[a,b]$

$$G'(\eta) = \frac{G(b)-G(a)}{b-a}$$

$$= 0 \quad \text{w/ } \eta \in (a,b)$$



$\Rightarrow G'(x)$ has $n+1$ distinct roots in (a,b)

$G''(x) \Rightarrow n$ distinct roots in (a,b)

$G'''(x) \Rightarrow n-1$ roots

$\vdots$

$G^{(n+1)}(x) \Rightarrow$ has one root in (a,b)

$\Rightarrow$ call it α

Recall

$$G(x) = \underbrace{e(x)}_{} - \frac{\psi(x)}{\psi(t)} e(t) \quad = \text{const}$$

$\psi(x)$, $\psi(t)$: polynomials of deg. $n+1$

$p_n(x) - f(x)$

deg n

$$0 = G^{(n+1)}(\alpha) = f^{(n+1)}(\alpha) - \left(\frac{\partial^{n+1}}{\partial x^{n+1}} \psi(x) \right) \frac{e(t)}{\psi(t)}$$

$$\psi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

$$= \underbrace{x^{n+1}}_{} + \underbrace{c_n x^n + c_{n-1} x^{n-1} \cdots}_{\text{vanish upon differentiating } n+1 \text{ times.}}$$

$\rightarrow (n+1)!$ after $n+1$ differentiations

$$0 = G^{(n+1)}(\alpha) = f^{(n+1)}(\alpha) - (n+1)!\,\frac{e(t)}{\psi(t)}$$

$$\Rightarrow \quad e(t) = \frac{f^{(n+1)}(\alpha)}{(n+1)!}\,\psi(t)$$

$$e(t) = f(t) - p_n(t)$$

$$= \frac{f^{(n+1)}(\alpha)}{(n+1)!}\,\underbrace{(t-x_0)(t-x_1)\cdots(t-x_n)}_{\text{ignore for now}}$$

If $f^{(n+1)}(x)$ are bounded, then as $n \to \infty$, $|e(t)| \to 0$

<u>Ex</u>: $f(x) = \dfrac{1}{1+ax^2}$, $f'(x) = \dfrac{-2ax}{(1+ax^2)^2}$

$$f''(x) = \frac{8a^2x^2}{(1+ax^2)^3} - \frac{2a}{(1+ax^2)^2}$$

$$\Rightarrow \quad f^{(n+1)}(x) = O(a^{n+1})$$

If $a = 10$, a^{n+2} = huge

<u>Ex</u>: $f(x) = \sin(x)$ on $[-2,2]$

$$\max_{[-2,2]} |f^{(n+1)}| = 1 \Rightarrow |e(t)| \leq \frac{\prod_{j=0}^{n}(t-x_j)}{(n+1)!}$$

$$\frac{\prod_{j=0}^{n}(t-x_j)}{(n+1)!} \leq \frac{4^{n+1}}{(n+1)!}$$ for equally spaced points

$$\frac{4^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow |e(t)| \leq \frac{4^{n+1}}{(n+1)!} \to 0$$

We have existence, uniqueness, & conditions on convergence as $n \to \infty$