Last time: solve polynomial interpolation problem by solving $Vc = y$

V = Vandermonde matrix

$$= \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots \\ 1 & x_1 & x_1^2 & \cdots \\ \vdots & \vdots & \vdots & \\ 1 & x_n & x_n^2 & \cdots \end{bmatrix}$$

c = coefficients of $p_n(x) = \sum_{i=0}^{n} c_i x^i$

$$y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_n) \end{bmatrix}$$

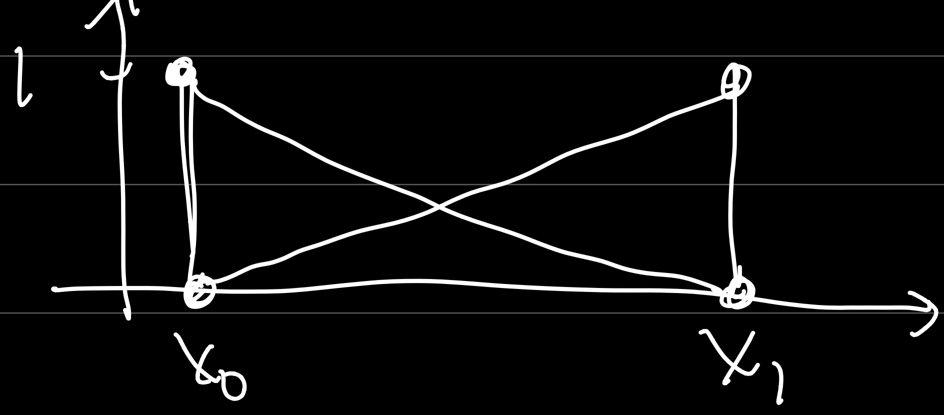
Find $p_n(x)$ st $p_n(x_i) = y_i \quad i = 0, \dots, n$

Lagrange & Newton bases

Ex: $\{x_0, x_1\}$, $\{y_0, y_1\}$

$$p(x) = y_0 \underbrace{\frac{(x - x_1)}{(x_0 - x_1)}}_{l_0(x)} + y_1 \underbrace{\frac{(x - x_0)}{(x_1 - x_0)}}_{l_1(x)}$$

$$p(x) = \sum_{i=0}^{n} y_i \, l_i(x)$$



<u>Higher n:</u> $$p_n(x) = \sum_{i=0}^{n} y_i \, l_i(x)$$

$$l_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\Rightarrow p_n(x_j) = \sum_{i=0}^{n} y_i \, \cancel{l_i(x_j)} = y_j$$

<u>Construction:</u> $$l_0(x) = \frac{(x-x_1)\cdots(x-x_n)}{(x_0-x_1)\cdots(x_0-x_n)}$$

$$l_0(x_0) = \frac{(x_0-x_1)\cdots(x_0-x_n)}{(x_0-x_1)\cdots(x_0-x_n)} = 1$$

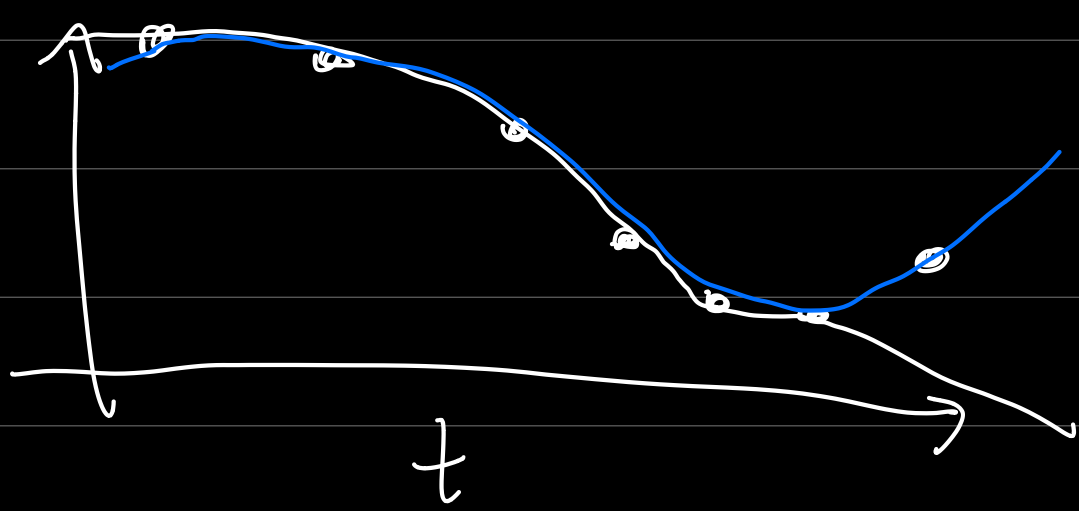
$$l_0(x_j) = 0$$

$$\Rightarrow \quad l_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}$$

Vandermonde matrix for Lagrange polynomials

$$V = \begin{bmatrix} l_0(x_0) & l_1(x_0) & \cdots \\ l_0(x_1) & l_1(x_1) & \cdots \\ l_0(x_2) & \vdots & \cdots \\ \vdots & \vdots & \end{bmatrix} = I$$

<u>Newton basis</u> : alternative to Lagrange



Can introduce a new x_n & y_n

without having to rebuild $p_n(x)$.

$$\underbrace{p_n(x)}_{\substack{\text{interp} \\ n+1 \text{ pts.}}} = \underbrace{p_{n-1}(x)}_{\substack{\text{interp. } n \\ \text{pts}}} + r_n(x), \quad r \in P^n$$

$$p_n(x) - p_{n-1}(x) = r_n(x)$$

$$\underbrace{p_n(x_i)}_{y_i} - \underbrace{p_{n-1}(x_i)}_{y_i} = r_n(x_i) \quad i = 0, \dots, n-1$$

$$\Rightarrow \quad r_n(x_i) = 0 \qquad i = 0, \dots, n-1$$

$$r_n(x) = c_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

$$r(x_n) = \underbrace{p_n(x_n)}_{f(x_n)} - p_{n-1}(x_n)$$

$$= c_n \prod_{j=0}^{n-1} (x_n - x_j)$$

$$q_n(x_n)$$

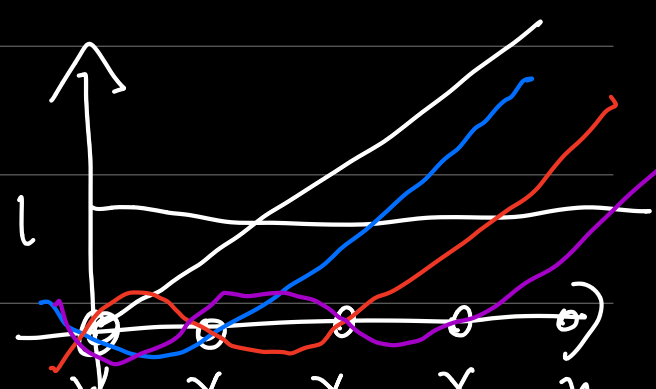
$$\boxed{c_n = \frac{f(x_n) - p_{n-1}(x_n)}{\prod_{j=0}^{n-1}(x_n - x_j)}}$$

$$\begin{aligned}
p_n(x) &= p_{n-1}(x) + r_n(x) \\
&= p_{n-1}(x) + c_n q_n(x) \\
&= p_{n-2}(x) + \underbrace{r_{n-1}(x)}_{c_{n-1} q_{n-1}(x)} + c_n q_n(x) \\
&= \underbrace{p_0(x)}_{c_0 q_0(x)} + c_1 q_1(x) + c_2 q_2(x) \cdots \\
&= \sum_{i=0}^{n} c_i q_i(x)
\end{aligned}$$

$$q_i(x) = \prod_{j=0}^{i-1} (x - x_i)$$

$$q_0(x) = 1$$

$$q_1(x) = x - x_0$$

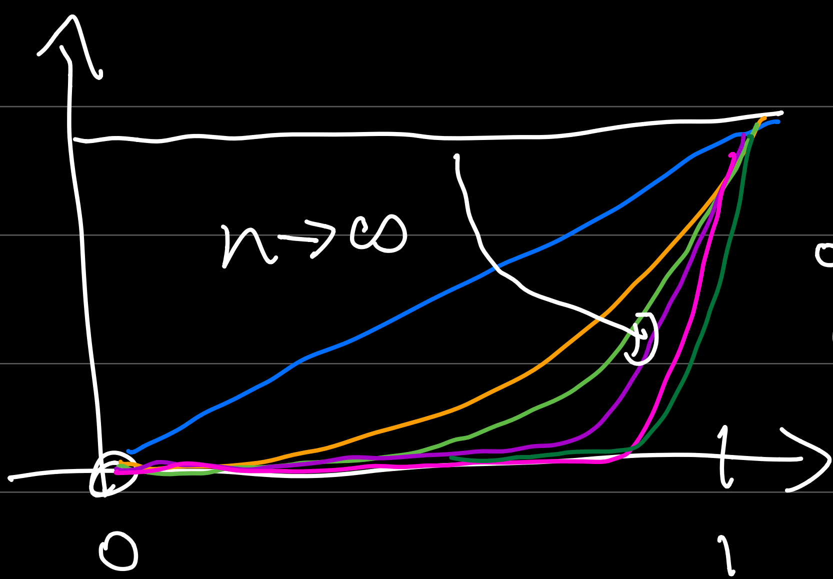


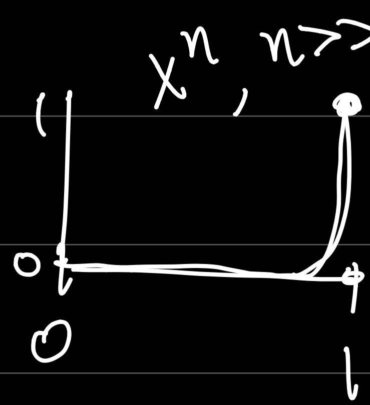
$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$$q_2(x) = (x - x_0)(x - x_1)$$

$$q_3(x) = (x - x_0)(x - x_1)(x - x_2)$$

Problems w/ monomials:





$$\Rightarrow \quad V = \begin{bmatrix} | & | & | & & | \\ x^0 & x^1 & x^2 & \cdots & x^n \\ | & | & | & & | \end{bmatrix}$$

$$\xrightarrow{\quad n \quad}$$

for n large, columns of V start to look similar.

$\Rightarrow$ as $n \to \infty$, $V \to$ a singular matrix.

$$p_n(x) = \sum_{i=0}^{n} c_i q_i(x)$$

$$p_n(x_i) = y_i \qquad i = 0, \dots, n$$

$$Vc = y$$

$$V = \begin{bmatrix} q_0(x_0) & q_1(x_0) & \cdots \\ q_0(x_1) & q_1(x_1) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$q_i(x) = \prod_{j=0}^{i-1} (x - x_j)$$

$$q_i(x_j) = 0 \qquad j = 0, \dots, i-1$$

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & q_1(x_1) & 0 & 0 \\ 1 & q_1(x_2) & q_2(x_2) & 0 \\ 1 & \vdots & & \end{bmatrix}$$

$\Rightarrow$ lower triangular is easy to solve using back-sub.