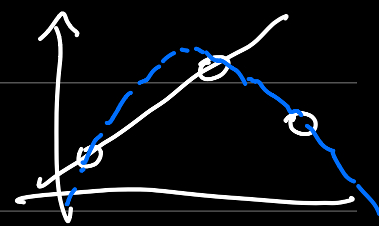
Last time: Newton bases for interpolation

$$p_n(x) = \sum_{i=0}^{n} c_i q_i(x)$$

$$q_i(x) = \prod_{j=0}^{i-1} (x - x_j)$$



$$\Rightarrow \quad c_n = \frac{f(x_n) - p_{n-1}(x_n)}{\prod_{j=0}^{n-1}(x_n - x_j)} \qquad p_{n-1}(x_n) \to \sum_{i=0}^{n-1} c_i q_i(x)$$

$$= \frac{f(x_n) - p_{n-1}(x_n)}{q_n(x_n)}$$

<u>Def</u>: $f[x_0] = f(x_0)$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$p_n(x) =$ Newton basis.

$$p_0(x) = c_0 q_0(x) = c_0$$

$$\Rightarrow \quad c_0 = f(x_0) = f[x_0]$$

$$p_1(x) = c_0 q_0(x) + c_1 q_1(x)$$
$$= f[x_0] + c_1 (x - x_0)$$

$$c_1 = \frac{f(x_1) - p_0(x_1)}{x_1 - x_0}$$
$$= \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$
$$= f[x_0, x_1]$$

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + c_2 q_2(x)$$
$$c_2 = \frac{f(x_2) - p_1(x_2)}{\prod_{j=0}^{1} (x_2 - x_j)}$$

$$p_1(x_2) = f[x_0] + f[x_0, x_1](x_2 - x_0)$$

$$\Rightarrow c_2 = \frac{f[x_2] - f[x_0] - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_1)(x_2 - x_0)}$$

$$= \frac{\overset{①}{f[x_2] - f[x_1]} + \overset{②}{f[x_1] - f[x_0]} - \overset{③}{f[x_0, x_1](x_2 - x_0)}}{(x_2 - x_1)(x_2 - x_0)}$$

$$① = \frac{f[x_2] - f[x_1]}{x_2 - x_1} \frac{1}{x_2 - x_0} = \frac{f[x_1, x_2]}{x_2 - x_0}$$

$$② \quad \frac{f[x_1] - f[x_0]}{x_1 - x_0}(x_1 - x_0) = f[x_0, x_1](x_1 - x_0)$$

$$② + ③ = \frac{f[x_0, x_1]\left((x_1 - \cancel{x_0}) - (x_2 - \cancel{x_0})\right)}{(x_2 - x_1)(x_2 - x_0)}$$

$$= \frac{f[x_0, x_1]\,\cancel{(x_2 - x_1)}}{\cancel{(x_2 - x_1)}(x_2 - x_0)} = \frac{f[x_0, x_1]}{x_2 - x_0}$$

$$\Rightarrow c_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} := f[x_0, x_1, x_2]$$

How to compute $f[x_0, \dots, x_k]$?

$\to$ Divided diff. table.

	$p_0(x)$	$p_1(x)$	$p_2(x)$
$x_0 \to$	$f[x_0]$		
		$\frac{f[x_1]-f[x_0]}{x_1-x_0}$	
$x_1 \to$	$f[x_1]$		$\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$
		$\frac{f[x_2]-f[x_1]}{x_2-x_1}$	
$x_2 \to$	$f[x_2]$		$\dots$
		$\dots$	
$x_3 \to$	$f[x_3]$		
		$\dots$	
$x_4 \to$	$f[x_4]$		

<u>Ex:</u> $f(x_0) = -3,\ f(x_1) = 2,\ f(x_2) = 0$

$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1$

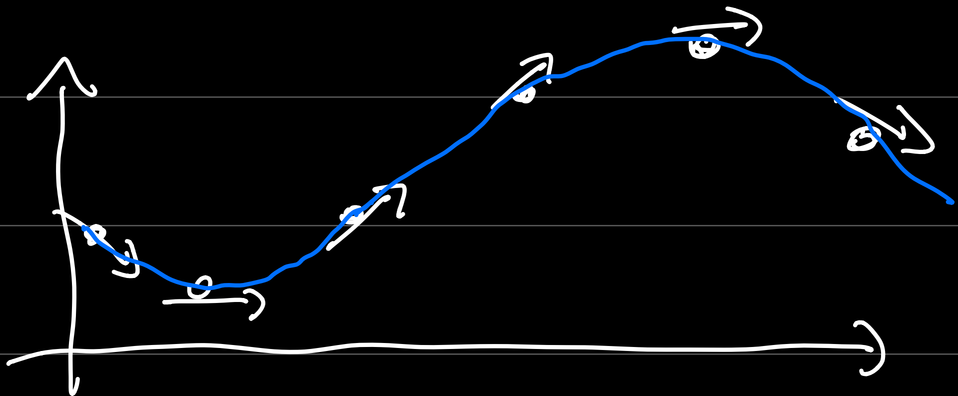
$f[x_0] = \boxed{-3}$			
	$\frac{2-(-3)}{0-(-1)} = \boxed{5}$		
$f[x_1] = 2$		$\frac{-2-5}{2} = \boxed{\frac{-7}{2}}$	
	$\frac{0-2}{1} = -2$		
$f[x_2] = 0$			

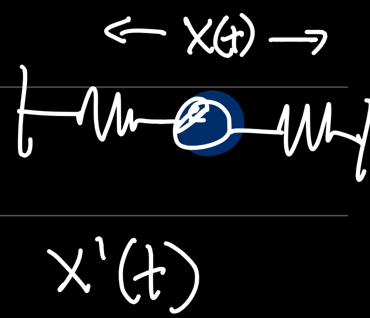
$$p_n(x) = -3 + 5(x+1) + \frac{-7}{2}(x+1)x$$

Check $p_n(0) = -3 + 5 = 2 = f(x_1 = 0)$

- Error analysis for interp.
- Diff. basis functions

$\Big\}\ p_n(x_i) = f(x_i)$





$\Rightarrow$ Hermite interpolation:

given $\{x_0, \dots, x_n\}$

$\{f(x_0), \dots, f(x_n)\}$

$+ \quad \{f'(x_0), \dots, f'(x_n)\}$

Find $p \in P^{2n+1}$ s.t.

$n+1 \quad \leftarrow p(x_i) = f(x_i)$ for $i = 0, \dots, n$

$n+1 \quad p'(x_i) = f'(x_i)$

Deg $2n+1 \Rightarrow 2n+2$ coeffs $= 2(n+1)$

① Is this solvable? Yes (& unique)

$\Rightarrow$ if pts distinct

$$P_{2n+1}(x) = \sum_{i=0}^{n} f(x_i) A_i(x) + \sum_{i=0}^{n} f'(x_i) B_i(x)$$

conditions satisfied if $\Rightarrow \begin{cases} A_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \\ A_i'(x_j) = 0 \end{cases}$

& $\begin{cases} B_i(x_j) = 0 \\ B_i'(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \end{cases}$

$$P_{2n+1}(x_j) = \sum f(x_i) A_i(x_j) = f(x_j)$$

$$P_{2n+1}'(x_j) = \underbrace{\sum f(x_i) A_i'(x_j)}_{0} + \underbrace{\sum f'(x_i) B_i'(x_j)}_{f'(x_j)}$$

$\Rightarrow$ Build $A_i(x)$, $B_i(x)$ on hw.

Recall $l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$

$A_i(x) = (1-2(x-x_i)l_i'(x_i))\,l_i^2(x)$

$B_i(x) = (x-x_i)\,l_i^2(x)$

$\Rightarrow$ Show properties on hw.

(MVT)

Uniqueness: use Rollé's / Mean Value Thm

$\Rightarrow$ if $f \in C^1[a,b]$ & $f(a) = f(b)$

$\exists\, c \in (a,b)$ s.t. $f'(c) = 0$

Thm: Hermite interp. is unique if $\{x_i\}_{i=0}^{n}$ are distinct + $n \geq 0$.

Pf: Assume $\exists\, p, q \in P^{2n+1}$ both Hermite interpolants. $w(x) = p(x) - q(x) \in P^{2n+1}$

$\Rightarrow w(x_i) = 0 \Rightarrow n+1$ roots.

By Rollé's, $w'(x)$ has n distinct roots

$\Rightarrow$ in (x_i, x_{i+1}), $w'(x)$ has a root

b/c $w(x_i) = w(x_{i+1}) = 0$ (n roots)

but $w'(x_i) = 0$ for $i = 0, \dots, n$

$\Rightarrow$ $w' \in P^{2n}$ has $= 2n+1$ zero roots

$\Rightarrow$ $w' = 0$ $\Rightarrow$ $w = \text{const.}$

But $w(x_i) = 0$ $\Rightarrow$ $w(x) = 0$