

## MODEL REDUCTION EXPERIMENTS PART II: MOMENT MATCHING REDUCTION

M. EMBREE AND D. C. SORENSEN\*

**Abstract.** This document describes a set of experiments for students in CAAM 651 at Rice University, Spring 2011, that aims to foster insight into the structure and behavior of a particular linear dynamical system. Students should complete these exercises for one course credit.

**0. Instructions.** After studying basic control theory (as addressed in project 1), our CAAM 651 lectures addressed moment matching algorithms for reducing single-input, single-output linear time invariant dynamical systems of the usual form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) &= \mathbf{c}\mathbf{x}(t) + du(t),\end{aligned}$$

with  $\mathbf{b} \in \mathbb{C}^{n \times 1}$ ,  $\mathbf{c} \in \mathbb{C}^{1 \times n}$ , and  $d \in \mathbb{C}^{1 \times 1}$ . In particular, we considered the Arnoldi, bi-Lanczos, and rational Krylov algorithms, which produce reduced-order models that match particular moments of the system at specified interpolation points. The exercises described here invite you to investigate how these algorithms perform on a real dynamical system.

This list of tasks is intentionally open-ended. The investigations you conduct will be guided by your own instinct and insight, along with any restrictions imposed by the size and properties of the matrix you select. Be frank if you are observing a phenomenon you cannot understand; feel free to include additional experiments that might give you greater insight. If the size of the matrix you have chosen prohibits you from completing some aspects of the assignment, please adapt the problem to something more reasonable for your system. We are happy to discuss strategies with you to help facilitate large-scale computations.

The codes required for this assignment are stored in the `MORcodes.zip` file on the class website.

**0.1. A note about your write-up.** Please submit a typed report describing the experiments you conduct, as directed by the tasks below. We recommend that you use the  $\text{\LaTeX}$  typesetting system to collect and organize your results. In particular, you can download a  $\text{\LaTeX}$  template, `modred_tasks2.tex`, from the class website, which you can then edit; this will give our reports a uniform appearance. (This document uses the SIAM journal macros. These macros and supporting files, along with all the graphics files used in this report, are collected in one `.zip` file on the class website).

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\*Department of Computational and Applied Mathematics, Rice University, 6100 Main Street—MS 134, Houston, Texas 77005–1892 ([embree@rice.edu](mailto:embree@rice.edu), [sorensen@rice.edu](mailto:sorensen@rice.edu)).

**1. Arnoldi reduction.** To begin with, select a system from your research, or from those listed on the course website. (If the system has multiple inputs (outputs), please just choose one column (row) of  $\mathbf{b}$  ( $\mathbf{c}$ ) to obtain a single-input, single-output system.

The basic Arnoldi algorithm (use `ArnoldiC.m`), run for  $k$  steps, produces a reduced model

$$\mathbf{A}_k = \mathbf{V}_k^* \mathbf{A} \mathbf{V}_k, \quad \mathbf{b}_k = \mathbf{V}_k^* \mathbf{b}, \quad \mathbf{c}_k = \mathbf{c} \mathbf{V}_k,$$

where the columns of  $\mathbf{V}_k \in \mathbb{C}^{n \times k}$  form an orthonormal basis for the Krylov subspace  $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ . This reduced model should match  $k$  moments of the system at the interpolation point  $s = \infty$ :

$$\mathbf{c} \mathbf{A}^j \mathbf{b} = \mathbf{c}_k \mathbf{A}_k^j \mathbf{b}_k, \quad j = 0, \dots, k-1.$$

Use the routine `ArnoldiC.m` to generate the matrix  $\mathbf{V}_k$ , and study various properties of the reduced order models for various values of  $k$ .

- 1.1. Verify that the first  $k$  moments of the system match, but the  $k+1$  moment does not match, for several modest values of  $k$ .
- 1.2. Are your reduced systems stable? Produce a plot showing the spectral abscissa of  $\mathbf{A}_k$  as a function of  $k$ .
- 1.3. For several values of  $k$ , use the `Mysigma_log.m` routine to produce a plot that compares the transfer function for the original system to the reduced system. (It is probably best to make one plot for each value of  $k$ , to reduce clutter.)
- 1.4. Plot the transfer function of the error system for several values of  $k$ , as described in Section 1.1 below.

**1.1. Plotting the error system.** Suppose you have used the Arnoldi algorithm (or any other method) to reduce your system to some  $k$ -dimensional state space, resulting in the system

$$\begin{aligned} \dot{\mathbf{x}}_k(t) &= \mathbf{A}_k \mathbf{x}_k(t) + \mathbf{b}_k u(t) \\ y(t) &= \mathbf{c}_k \mathbf{x}_k(t) + d u(t). \end{aligned}$$

We can examine the error between the original and reduced models,

$$e(t) := y(t) - y_k(t),$$

as the output of an augmented system. In particular, notice that we can combine the two state-space differential equations to get

$$\frac{d}{dt} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_k(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_k \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_k(t) \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{b}_k \end{bmatrix} u(t),$$

so the error is then the output

$$e(t) = [\mathbf{c} \quad -\mathbf{c}_k] \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_k(t) \end{bmatrix}.$$

Thus, one can plot a Bode diagram for the error by applying the `Mysigma_log.m` routine to this new augmented system.

**2. Bi-Lanczos reduction.** How do your results change if you use the bi-Lanczos method, rather than the Arnoldi reduction? Use the `NSLanczos.m` code to compute matrices  $\mathbf{V}_k$  and  $\mathbf{W}_k$ , whose columns form bi-orthogonal bases for  $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$  and  $\mathcal{K}_k(\mathbf{A}^*, \mathbf{c}^*)$ , and then set

$$\mathbf{A}_k = \mathbf{W}_k^* \mathbf{A} \mathbf{V}_k, \quad \mathbf{b}_k = \mathbf{W}_k^* \mathbf{b}, \quad \mathbf{c}_k = \mathbf{c} \mathbf{V}_k,$$

where the columns of  $\mathbf{V}_k \in \mathbb{C}^{n \times k}$  form an orthonormal basis for the Krylov subspace  $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ . This reduced model should match  $2k$  moments of the system at the interpolation point  $s = \infty$ :

$$\mathbf{c}^* \mathbf{A}^j \mathbf{b} = \mathbf{c}_k^* \mathbf{A}_k^j \mathbf{b}_k, \quad j = 0, \dots, 2k - 1.$$

- 2.1. Verify that the first  $2k$  moments of the system match, but the  $2k + 1$  moment does not match, for several modest values of  $k$ .
- 2.2. Are your reduced systems stable? Produce a plot showing the spectral abscissa of  $\mathbf{A}_k$  as a function of  $k$ .
- 2.3. For several values of  $k$ , use the `Mysigma_log.m` routine to produce a plot that compares the transfer function for the original system to the reduced system. (It is probably best to make one plot for each value of  $k$ , to reduce clutter.)
- 2.4. Plot the transfer function of the error system for several values of  $k$ , as described in Section 1.1.

Consider the results of these experiments in contrast to the Arnoldi reduction.

- 2.5. Was it more or less difficult to obtain a stable model with the bi-Lanczos approach? How do the errors compare for the same fixed degree  $k$ ?

**3. Shift-Invert Arnoldi reduction.** Now repeat these experiments for the shift-invert Arnoldi method, where the orthonormal columns of  $\mathbf{V}_k$  span  $\mathcal{K}_k(\mathbf{A}^{-1}, \mathbf{b})$ . (You can edit the `ArnoldiC.m` code so the matrix-vector product operation with  $\mathbf{A}$  ( $\mathbf{A} \star \mathbf{v}$ ) is replaced by a linear solve ( $\mathbf{A} \backslash \mathbf{v}$ ). The resulting reduced-order model

$$\mathbf{A}_k = \mathbf{V}_k^* \mathbf{A} \mathbf{V}_k, \quad \mathbf{b}_k = \mathbf{V}_k^* \mathbf{b}, \quad \mathbf{c}_k = \mathbf{c} \mathbf{V}_k$$

should match  $k$  moments of the system at the interpolation point  $s = 0$ :

$$\mathbf{c} \mathbf{A}^{-j} \mathbf{b} = \mathbf{c}_k \mathbf{A}_k^{-j} \mathbf{b}_k, \quad j = 0, \dots, k - 1.$$

Repeat the above experiments with this new reduction.

- 3.1. Verify that the first  $k$  moments of the system match, but the  $k + 1$  moment does not match, for several modest values of  $k$ .
- 3.2. Are your reduced systems stable? Produce a plot showing the spectral abscissa of  $\mathbf{A}_k$  as a function of  $k$ .
- 3.3. For several values of  $k$ , use the `Mysigma_log.m` routine to produce a plot that compares the transfer function for the original system to the reduced system. (It is probably best to make one plot for each value of  $k$ , to reduce clutter.)
- 3.4. Plot the transfer function of the error system for several values of  $k$ , as described in Section 1.1 below.

Based on these experiments, answer the following qualitative question.

- 3.5. How does your error qualitatively differ from the plots generated by the Arnoldi reduction in Question 1?

**4. Rational Krylov reduction.** The routine `IRKA.m` can be used generate orthonormal bases for rational Krylov spaces, which allow for the construction of reduced models that match moments at a number of interpolation points; see Theorem 2.5 in the course notes for explicit details.

- 4.1. Conduct a few experiments with the Iterative Rational Krylov Algorithm (implemented in `IrkaN.m` using various tolerances. Plot the transfer function for the error system. Can you do better than Arnoldi and shift-invert Arnoldi for the same fixed degree  $k$ ?

Optional: Adapt the `IrkaN.m` code to print out the shifts that are automatically determined by the algorithm. Show how these evolve from iteration to iteration.