CAAM 652: Homework 1

1. Prove that the DG method constructed in class for the acoustic wave equation is stable under periodic boundary conditions. Hint: multiply the DG system on the left by p and u, sum the two equations up, and manipulate the integrals to cancel all spatial terms.

2. The code `advection_convergence.m` uses an upwind flux

$$\frac{a}{2}[\![u]\!] - \tau\frac{|a|}{2}[\![u]\!], \qquad \tau > 0$$

   with $\tau = 1$. Modify the convergence code to run with a central flux $\frac{a}{2}[\![u]\!]n$ (i.e. $\tau = 0$), and plot L^2 errors for $N = 1, \ldots, 4$. Describe how the error behaves differently compared to the $\tau = 1$ case.

3. The code computes only right hand sides of an ODE system

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} = -\boldsymbol{A}\boldsymbol{u}. \tag{1}$$

   It does so by efficiently evaluating the matrix-vector product $-\boldsymbol{A}\boldsymbol{u}$.

   Applying the right hand side to the canonical vectors $\boldsymbol{e}_i$ results in the ith column of the matrix. Use this to build the global matrix form of the semi-discrete system for the periodic advection equation with $a = 1$. For $N = 4$ and $K = 16$, compute and plot the eigenvalues of $-\boldsymbol{A}$ for $\tau = 0, .5, 1, 2$. You should see eigenvalues with negative real parts; since the exact solution to (1) is $\boldsymbol{u} = e^{-\boldsymbol{A}t}$, this will imply that certain components of the solution are *damped*.

   Based on the computed eigenvalues, how would increasing τ larger and larger affect the application of explicit time-steppers to (1)? (Hint: consider regions of stability).