A mixed finite element approach for viscoelastic wave propagation

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Abstract

In this paper, we are interested in the modeling of wave propagation in viscoelastic media. We present a family of models which generalize the Zener's model. We achieve its mathematical analysis: existence and uniqueness of solutions, energy decay and propagation with finite speed. For the numerical resolution, we extend a mixed finite element method proposed in [8]. This method combines mass lumping with a centered explicit scheme for time discretization. For the resulting scheme, we prove a discrete energy decay result and provide a sufficient stability condition. For the numerical simulation in open domains we adapt the perfectly matched layers techniques to viscoelastic waves [23]. Various numerical results are presented.

Keywords: Energy dissipation, finite velocity propagation, mixed finite element, stability analysis, viscoelastic waves, Zener's model.

1 Introduction

For the numerical simulation of wave propagation in solids, particularly for the applications in geophysics, it is now commonly admitted that it is important to take into account the attenuation effects due to the visco-elastic nature of the medium [12, 15]. One of the first problem that one meets is the choice of the appropriate visco-elasticity model, from the wide variety of models provided by the physical literature [10, 20, 29, 42]. The generalized Zener's models, that we shall consider in this paper, has the advantage to be presented in a unified framework (that we shall describe in section 2) and to offer a sufficiently large flexibility to permit to take into account most of the interesting phenomena from the geophysical point of view.

The numerical treatment of such media is already a rather old subject since it began about 15 years ago (see for instance the works of Carcione et al. ([14, 16, 17]). Most of the methods developed up to now are based on rather simple finite difference methods [11, 19, 38]. In this spirit, one reference work is due to Blanch, Robertson and Symes [41] who achieved in particular a stability and accuracy analysis of their method. Note however that their analysis, based on the Fourier method, is restricted to 1D homogeneous media. It is known that a robust way to treat heterogeneous media is to use finite element methods. In the context of visco-elastic waves, it seems that there are very few existing works, in particular of mathematical nature. Let us mention however the work done by Janovský, Shaw, Warby and Whiteman [36] in which the authors propose a finite element method in space and a quadrature rules in time to solve an isotropic viscoelastic problem based on a formulation in displacement and an integral representation of the viscoelastic model (see also [43] for quasi-static problems), or more recently (of much less mathematical nature) by Isedman, Niekamp and Stein [35] about space-time finite elements. Let us finally emphasize the work of Ha, Santos and Sheen [32] where they are interested in the nonconforming finite element approximation of a viscoelastic complex model in the frequency domain, which makes possible to allow the coefficients of the model to depend on ω. Recently, in [8], we develop a new mixed finite element method for elastodynamics equations. This method is especially designed for regular meshes but presents

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the interest to allow the coupling with the fictitious domain method to treat complicated geometries of the propagation domain [7, 21]. Our objective in this paper is essentially to extend (and analyze) this method to (a class of) visco-elastic media. More precisely, there are two main parts in this article:

- Provide a rather complete mathematical theory for the generalized Zener's model. This is the object of section 3. The main interest of this section is to be preparatory to the numerical analysis of section 4 and we do not pretend that neither the results, nor the method we use to prove them, are very original. In fact, the mathematical analysis of wave propagation in viscoelastic media has been initiated about forty years ago. The basic general theory was developed in particular by Gurtin and Sternberg [31] and first existence and uniqueness results, which however can not be applied to the models that we consider here, are due to Duvaut and Lions [25]. Concerning Zener's model, in dimension 1, we must cite the PhD thesis of G. Canadas [30]. Since then, a lot of progress has been accomplished and, rather recently, a complete monograph has been written by Fabrizio and Morro [27] on the mathematical problems of linear visco-elasticity. Their results, very general, are essentially based on the use of Laplace transform methods and are applicable to a large class of models that include generalized Zener's models. However, it is not clear that our results of section 3 are contained in [27] (for instance, we use a different approach based on semi-group theory). In particular, the results concerning energy decay (theorem 3.3) and finite velocity propagation (theorem 3.5) seem to be new.
- Construct a numerical method that generalizes the one of [8] and guarantees the explicit nature of the numerical scheme as well as its stability even in the case of heterogeneous media (section 4). This can be achieved by working with the stress-displacement formulation of the propagation equations (we used the velocity-stress formulation in [8]), by using mass lumping as in [8] (this is one of the main interest of the new finite element) and - this is the main point - by applying a specific time stepping of the constitutive law. The stability analysis is based on a discrete energy decay result (theorem 4.1) that mimics the continuous result of theorem 3.3. We obtain a sufficient stability condition (theorem 4.2) that coincides in the 1D homogeneous case with the stability condition that can be obtained by the Fourier method. In this sense, our analysis generalizes the one by Blanch, Robertson and Symes [41].

Sections 3 and 4 are the two main sections of this article. They are preceded, in section 2, by a brief recap of visco-elasticity theory with a more detailed presentation of generalized Zener's models that are the object of our study, In section 5, we shall present various numerical results obtained with our method. This section includes in particular a description of the way we generalized the Perfectly Matched Layers method (a method to treat transparent boundaries [9, 23]) to viscoelastic waves (subsection 5.1) and a numerical simulation in a "realistic" model for which we have designed specifically the parameters of our Zener's model in order to realize a quasi Q-constant model (subsection 5.2.3).

2 Viscoelastic models

2.1 The general models

The linear viscoelastic models [29] take into account the waves absorption phenomenon, in such models if we consider the strain tensor at time t:

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

associated to a displacement field $u(x,t)$ ($x \in \Omega \subset \mathbb{R}^n$, $n = 1,2,3$), the linear viscoelastic material to be one fore which the stress tensor σ is related to ε by a convolution integral [20, 29, 42] as follows:

$$\sigma_{ij}(x,t) = \int_{-\infty}^{t} G_{ijkl}(t-\tau)\frac{\partial \varepsilon_{kl}(\tau)}{\partial \tau}\, d\tau, \tag{2}$$

where G is a tensor of order 4, symmetric:

$$G_{ijkl} = G_{jikl} = G_{ijlk}. \tag{3}$$