MATH 213 Spring 2008 Exam 1. 5 problems in 14 parts. 5 points per part.

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1. [5 points] Solve $x_{n+1} = x_n/10$ subject to $x_0 = 100$.

2. [5 points] Solve $x_{n+1} = x_n/10 + 9/10$ subject to $x_0 = 100$.

3. Consider the pair of first order equations

$$\begin{aligned} x_{n+1} &= 2x_n - y_n \\ y_{n+1} &= -x_n + 2y_n \end{aligned} \tag{1}$$

(i) [5 points] Show that x_n obeys the second order equation

$$x_{n+2} = 4x_{n+1} - 3x_n \tag{2}$$

(ii) [5 points] Insert the guess $x_n = c\lambda^n$ into equation (2) and find the two roots $\lambda_\pm$

(iii) [5 points] The general solution is now $x_n = c_+\lambda_+^n + c_-\lambda_-^n$. Find $c_\pm$ when $x_0 = 1$ and $y_0 = 0$.

(iv) [5 points] Write equation (1) as a matrix equation $V_{n+1} = MV_n$. What is M?

(v) [5 points] Solve $MV = \lambda V$ for the two eigenvalues, $\lambda_\pm$, and associated eigenvectors, $V_\pm$.

(vi) [5 points] The general solution is now $V_n = c_+\lambda_+^n V_+ + c_-\lambda_-^n V_-$. Find $c_\pm$ when $x_0 = 1$ and $y_0 = 0$.

4. Suppose a, b and c are positive and consider

$$x_{n+1} = f(x_n) \quad \text{where} \quad f(x) = \frac{cx}{1 + (ax)^b}$$

(i) [5 points] Graph f and find and label its two steady states. Under what condition on c are both such states nonnegative?

(ii) [5 points] Show that the nonzero steady state is stable so long as $b < 2c/(c-1)$.

(iii) [5 points] Produce a careful cobweb sketch in the case that $b = 1$.

5. Suppose a, b, c and d are positive and consider

$$\begin{aligned} x_{n+1} &= ax_n - bx_n y_n \\ y_{n+1} &= dx_n y_n - cy_n \end{aligned} \tag{3}$$

(i) [5 points] Find the two steady states. Under what condition on a are both such states nonnegative?

(ii) [5 points] Compute the matrix of partial derivatives of equation (3) and evaluate it at the nonzero steady state. Your final result should be

$$J = \begin{pmatrix} 1 & -b(1+c)/d \\ d(a-1)/b & 1 \end{pmatrix}$$

(iii) [5 points] Compute the eigenvalues of J and argue that at least of them lies outside the unit circle.