Numerical Methods for Ordinary Differential Equations

We will develop and test three approaches on the simple planar system

\[
\begin{align*}
x_1' &= x_1 - x_1^3 - ax_2 \\
x_2' &= x_1 - x_2 \\
\end{align*}
\]  \hspace{1cm} (1)

We begin with the canned solver \texttt{ode23}. In order to solve (1) from \(t = 0\) up to \(t = 20\) starting from \(x_1(0) = x_2(0) = 1\) we type

\[
[t, x] = \texttt{ode23(@mycub, [0 20], [1 1])}
\]

after creating an m-file called \texttt{mycub.m} that looks like,

\[
\begin{align*}
\text{function } dx &= \text{mycub}(t, x) \\
\text{a} &= 2; \\
dx(1,1) &= x(1) - x(1)^3 - a \times x(2); \\
dx(2,1) &= x(1) - x(2); \\
\end{align*}
\]

One may then examine \(x\) in the phase plane via

\[
\text{plot(x(:,1),x(:,2)) or comet(x(:,1),x(:,2))}
\]

or as natural time functions via

\[
\text{plot(t,x(:,1),t,x(:,2))}
\]

and in each case one should label the axes and curves via \texttt{xlabel}, \texttt{ylabel} and \texttt{legend}. One may now experiment with different time spans, initial conditions, and parameters (just \(a\) for now). As a second example we encode the molecular switch in equation 29 on page 290 via

\[
\begin{align*}
\text{function } dx &= \text{switch29}(t, x) \\
dx(1,1) &= x(1) - x(1)^2 - 2 * x(1) * x(2); \\
dx(2,1) &= x(2) - x(2)^2 - 2 * x(1) * x(2); \\
\end{align*}
\]

To better appreciate what \texttt{ode23} is up to we now embark on our own approximation scheme. It is really just a step back to the early part of the course, for we will replace the differential equation, (1), with the difference equation

\[
\begin{align*}
\frac{x_1(t_{n+1}) - x_1(t_n)}{dt} &= x_1(t_n) - x_1^3(t_n) - ax_2(t_n) \\
\frac{x_2(t_{n+1}) - x_2(t_n)}{dt} &= x_1(t_n) - x_2(t_n) \\
\end{align*}
\]  \hspace{1cm} (2)
where \( dt \) is our time step, or increment, and \( t_n = (n-1)dt \). A slight rearrangement of (2) reveals that we may solve for the state at \( t_{n+1} \) knowing only the state at \( t_n \). That is

\[
\begin{align*}
  x_1(t_{n+1}) &= x_1(t_n) + dt(x_1(t_n) - x_1^3(t_n) - ax_2(t_n)) \\
  x_2(t_{n+1}) &= x_2(t_n) + dt(x_1(t_n) - x_2(t_n))
\end{align*}
\]

we have coded this in \texttt{mycub_fe.m}.