EXTREMAL EIGENVALUE PROBLEMS FOR COMPOSITE MEMBRANES, I

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Abstract: Given an open bounded connected set $\Omega \subset \mathbf{R}^N$ and a prescribed amount of two homogeneous materials of different density, for small k we characterize the distribution of the two materials in Ω that extremizes the kth eigenvalue of the resulting clamped membrane. We show that these extremizers vary continuously with the proportion of the two constituents. The characterization of the extremizers in terms of level sets of associated eigenfunctions provides geometric information on their respective interfaces. Each of these results generalizes to N dimensions the now classical one dimensional work of M.G. Krein.

1. INTRODUCTION

Within the class of fixed endpoint strings on the interval $(0,1)$ with density between α and β and mass equal to $\alpha\gamma+\beta(1-\gamma)$, M.G. Krein [19] was able to isolate those with the largest or smallest kth natural frequency. More precisely, denoting by $\lambda_k(\rho)$ the kth Dirichlet eigenvalue of the string with density ρ, and by $\check{\rho}_k^\gamma$ $(\hat{\rho}_k^\gamma)$ that function which minimizes (maximizes) $\rho \mapsto \lambda_k(\rho)$ over

$$ad_\gamma^* \equiv \{\rho \in L^\infty(0,1) : \alpha \le \rho(x) \le \beta,\ \int_0^1 \rho\, dx = \alpha\gamma + \beta(1-\gamma)\},$$

Krein [19] found the $\check{\rho}_k^\gamma$ and $\hat{\rho}_k^\gamma$ to be uniquely defined by,

$$\begin{aligned} \check{\rho}_k^\gamma(x) &= \begin{cases} \beta, & \text{if } x \in (m_k^j - \frac{1-\gamma}{2k}, m_k^j + \frac{1-\gamma}{2k}) \quad j=1,\ldots,k; \\ \alpha, & \text{otherwise.} \end{cases} \\ \hat{\rho}_k^\gamma(x) &= \begin{cases} \alpha, & \text{if } x \in (m_k^j - \frac{\gamma}{2k}, m_k^j + \frac{\gamma}{2k}) \quad j=1,\ldots,k; \\ \beta, & \text{otherwise.} \end{cases} \end{aligned} \tag{1.1}$$

Where $m_k^j = \frac{2j-1}{2k}$ is the midpoint of the interval $(\frac{j-1}{k}, \frac{j}{k})$.

We see that the extremizers are, in a sense, 'bang bang' controls. For on part of the string they take the minimum density, α, while on the remainder the maximum density, β. When $k=1$ this result follows from the fact that the first eigenfunction is a concave function vanishing at 0 and 1. For recalling that

$$\lambda_1(\rho) = \inf_{u \in W_0^{1,2}} \frac{\int_0^1 (u')^2\, dx}{\int_0^1 \rho u^2\, dx},$$

we see that in order to minimize (maximize) λ_1 one should place the more dense material where the string displacement is greatest (least), leaving the less dense material in the remainder. Hence, the interface between the two materials in the extremizing case occurs on a level set of the corresponding first eigenfunction. Krein's result for larger k follows upon minimizing (maximizing) λ_1 over each of the nodal domains associated with the extremal λ_k.

Although Krein's arguments are strictly one dimensional, his results remain plausible in arbitrary dimensions. That is, over a class of densities analogous to ad_γ^*, the fundamental tone of an N dimensional membrane is minimized (maximized) by placing the more dense material where the membrane displacement is greatest (least), and the less dense material in the remainder. This two part study is devoted to a proof of this observation and a discussion of a number of its consequences. We first situate this problem in a larger class of optimization problems.

Our motivation for an attack on this N dimensional problem of Krein comes from optimal design. Here, within a class of structures one selects that structure that extremizes a particular cost function. The class of structures that will concern us are those membranes on Ω with densities in

$$\begin{aligned} ad_\gamma = \{\alpha\chi + \beta(1-\chi) : &\ \chi \text{ is the characteristic function} \\ &\text{of a subset of } \Omega \text{ with measure } \gamma\}. \end{aligned}$$

These are the composite membranes with material of density α in volume fraction $\gamma/|\Omega|$ and material of density β in volume fraction $1-\gamma/|\Omega|$. The cost of a membrane with density $\rho \in ad_\gamma$ will be measured in terms of $\lambda_k(\rho)$, the membrane's kth eigenvalue. Our problem is then one of optimal design with respect to structural eigenvalues, an engineering discipline whose more than 160 pre-1980 works were surveyed by Olhoff [21]. The contributions cited by Olhoff, and those in this field that have followed, interpret optimal design in the sense of shape optimization. That is, their class of admissible structures consists of homogeneous bodies, typically beams and shafts, with varying cross sectional geometries. Our class of composite structures, and their subsequent analysis, is more in line with the rigorous attack on the problems of optimal design exemplified by Kohn and Strang [18].

Part I of this study establishes a topology for ad_γ and investigates the continuity of $\rho \mapsto \lambda_k(\rho)$ on the associated closure of ad_γ. Existence of extremizers for λ_k then follows immediately. The proof, via epi convergence, that these extremizers vary continuously with γ, the amount of α material, closes Part I. In Part II we demonstrate that much of (1.1) survives the passage to higher dimensions. In particular we characterize the extremizers for small k in terms of level sets of associated eigenfunctions. This characterization allows us to write the linear membrane equation in the form of a nonlinear eigenvalue problem. From this we deduce results on the geometry of the level sets of eigenfunctions, and subsequently on the geometry of the extremizers of $\rho \mapsto \lambda_1(\rho)$.

2. THE COMPOSITE MEMBRANE

By membrane we refer to a structure which occupies an open bounded connected set $\Omega \subset \mathbf{R}^N$ and whose material properties can be described at each point strictly in terms of a mass density $\rho : \Omega \to [0, \infty)$. In this section we formulate and study a class of densities giving rise to composite membranes. We begin with two homogeneous materials, with densities α and β. Taking $0 < \alpha < \beta$, a composite membrane simply corresponds to the choice of some measurable set $\omega \subseteq \Omega$, in the sense that we agree to place the material with density α in ω and that with density β in the complement $\Omega \setminus \omega$. In this way we arrive at the following set of admissible densities,

Definition 2.1. *ad is the set of all functions of the form $\alpha\chi + \beta(1 - \chi)$, where χ is the characteristic function of a measurable subset of Ω.*

In order to establish existence of optimal designs we shall need some compactness of the admissible set. As ad is not compact in any reasonable topology we look to its position in the weak* topology of $L^\infty(\Omega)$.

Proposition 2.2. *The weak* closure of ad is the convex weak* sequentially compact set $ad^* = \{\rho \in L^\infty(\Omega) : \alpha \leq \rho(x) \leq \beta$ a.e. in $\Omega\}$.*

Proof: That ad^* is the weak* closure of ad follows from a now standard argument in homogenization, for details see Tartar [25] Theorem 3. Homogenization provides the construction necessary to show that ad is weak* dense in the simple functions between α and β. The result then follows since these simple functions are strongly, hence weak*, dense in ad^*. The convexity and compactness are straightforward. ∎

In most situations of real interest one must work under the constraint that admissible composites may only contain the constituents in some fixed proportion. This translates into the following restriction on ad.

Definition 2.3. *ad_γ is the set of all functions of the form $\alpha\chi + \beta(1-\chi)$ where χ is the characteristic function of a subset of Ω of measure γ.*

The mass of the membrane being a weak* continuous function of density, we find

Proposition 2.4. *The weak* closure of ad_γ is the convex weak* sequentially compact set*

$$ad_\gamma^* = \{\rho \in L^\infty(\Omega) : \alpha \le \rho(x) \le \beta \text{ a.e. } in\ \Omega,\ \int_\Omega \rho\, dx = \alpha\gamma + \beta(|\Omega| - \gamma)\}.$$

We turn to a geometric characterization of the relation between ad_γ and ad_γ^*. Recall that $v \in K$ is an **extreme point** of the convex set K whenever $K \setminus v$ is convex. From Friedland [10] Lemma 1 we deduce,

Proposition 2.5. *The set of extreme points of ad_γ^* is precisely ad_γ.*

We now consider the map $\gamma \mapsto ad_\gamma^*$, an important component in the stability analyses of §5. This issue was raised by Kohn and Strang [18] in the context of their 'Localization Principle', regarding the more general problem of the G-convergence of conductivity tensors. Nonetheless, arguments analogous to ours serve to make their remarks precise. What is needed in either case is the notion of set convergence due to Kuratowski.

Let (V, τ) denote a point set V endowed with the topology τ. When τ is first countable and $\{K^n\}$ is a sequence of subsets of V then we may define respectively, the τ limit inferior and superior of the K^n,

$$\tau\text{–}\mathrm{Li}\, K^n = \{v \in V : \exists \{v^n\} \text{ s.t. } v^n \in K^n \text{ and } v^n \xrightarrow{\tau} v\},$$
$$\tau\text{–}\mathrm{Ls}\, K^n = \{v \in V : \exists \{K^{n_j}\} \subset \{K^n\} \text{ and } v^j \in K^{n_j} \text{ s.t. } v^j \xrightarrow{\tau} v\}.$$

Definition 2.6. *A sequence of subsets $\{K^n\}$ of a first countable space (V, τ) is said to converge in the sense of Kuratowski to the set K when*

$$K = \tau\text{–}\mathrm{Li}\, K^n = \tau\text{–}\mathrm{Ls}\, K^n. \tag{2.1}$$

When (2.1) holds we write $K = \tau\text{–}\mathrm{Lim}\, K^n$. The lower and upper set limits always satisfy

Proposition 2.7. *(Attouch [1] 1.33) $\tau\text{–}\mathrm{Li}\, K^n$ and $\tau\text{–}\mathrm{Ls}\, K^n$ are closed subsets of (V, τ) and $\tau\text{–}\mathrm{Li}\, K^n \subseteq \tau\text{–}\mathrm{Ls}\, K^n$.*

Abbreviating weak* by w^*, and denoting weak* convergence by $\stackrel{*}{\rightharpoonup}$, we find

Proposition 2.8. *If $\gamma_n \to \gamma$ then $ad^*_\gamma = w^*\text{–}\mathrm{Lim}\, ad^*_{\gamma_n}$ in $L^\infty(\Omega)$.*

Proof: We first show that $ad^*_\gamma \subseteq w^*\text{–}\mathrm{Li}\, ad^*_{\gamma_n}$. If $\rho \in ad_\gamma$ then there exists some $\Omega^\gamma \subset \Omega$ such that

$$\rho(x) = \begin{cases} \alpha & x \in \Omega^\gamma \\ \beta & x \in \Omega \setminus \Omega^\gamma. \end{cases}$$

Now choose an $\Omega_n \subseteq \Omega$ such that $|\Omega_n| = \gamma_n$ and $|\Omega_n \triangle \Omega^\gamma| = |\gamma - \gamma_n|$, where $\triangle$ denotes symmetric difference, and define

$$\rho^n(x) = \begin{cases} \alpha & x \in \Omega_n \\ \beta & x \in \Omega \setminus \Omega_n. \end{cases}$$

With this definition we see that $\rho^n \in ad^*_{\gamma_n}$ and

$$\left|\int_\Omega \rho^n \phi\, dx - \int_\Omega \rho\phi\, dx\right| \leq (\beta-\alpha)\int_{\Omega_n \triangle \Omega^\gamma} |\phi|\, dx \to 0, \qquad \forall\, \phi \in L^1(\Omega),$$

i.e., $\rho^n \stackrel{*}{\rightharpoonup} \rho$. This demonstrates that $ad_\gamma \subseteq w^*{-}\mathrm{Li}\, ad^*_{\gamma_n}$, which, together with the fact that $w^*{-}\mathrm{Li}\, ad^*_{\gamma_n}$ is weak* closed (Proposition 2.7), implies that $ad^*_\gamma \subseteq w^*{-}\mathrm{Li}\, ad^*_{\gamma_n}$.

Now choose a $\rho \in w^*{-}\mathrm{Ls}\, ad^*_{\gamma_n}$. By definition there exists a sequence $\{\rho^j\} \subset ad^*$ where (i) $\rho^j \in ad^*_{\gamma_{n_j}}$ and (ii) $\rho^j \stackrel{*}{\rightharpoonup} \rho$. From (i) and the fact that $\gamma_n \to \gamma$ we deduce

$$\int_\Omega \rho^j\, dx = \alpha\gamma_{n_j} + \beta(|\Omega| - \gamma_{n_j}) \to \alpha\gamma + \beta(|\Omega| - \gamma), \tag{2.2}$$

while from (ii) we infer that $\int_\Omega \rho^j\, dx \to \int_\Omega \rho\, dx$. Reconciling this with (2.2) we find $\rho \in ad^*_\gamma$ and conclude that $w^*{-}\mathrm{Ls}\, ad^*_{\gamma_n} \subseteq ad^*_\gamma$. Noting Proposition 2.7, we have shown $ad^*_\gamma \subseteq w^*{-}\mathrm{Li}\, ad^*_{\gamma_n} \subseteq w^*{-}\mathrm{Ls}\, ad^*_{\gamma_n} \subseteq ad^*_\gamma$. ∎

3. THE EIGENVALUE PROBLEM

The displacement, u, of a clamped membrane with density ρ on the bounded domain Ω from its rest position, satisfies

$$-\Delta u = \lambda\rho u \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{3.1}$$

A function $u \in W_0^{1,2}(\Omega)$ is a **weak** solution of (3.1) if for some $\lambda \in \mathbf{R}$

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \lambda \int_\Omega \rho u v\, dx \qquad \forall\, v \in W_0^{1,2}(\Omega).$$

A λ for which there exists a nontrivial weak solution to (3.1) will be called an **eigenvalue** of (3.1). The linear space of weak solutions associated with λ will be called its **eigenspace**, nontrivial elements of which shall be referred to as **eigenfunctions**. The **multiplicity** of an eigenvalue will be understood to be the dimension of its eigenspace. An eigenvalue whose multiplicity is one will be called **simple**. Denoting by J the imbedding of $W_0^{1,2}(\Omega)$ in its dual and by M_ρ the mapping $\rho \mapsto \rho u$, the existence of the eigenvalues

$$0 < \lambda_1(\rho) \leq \lambda_2(\rho) \leq \cdots \uparrow +\infty,$$

repeated according to multiplicity, follows readily from the compactness of $\mathcal{G}_\rho \equiv (-\Delta)^{-1} J M_\rho$. The properties of these eigenvalues and their associated eigenfunctions can best be deduced from the following variational characterization, for details see Mikhlin [20] §34. Corresponding to the $\{\lambda_k(\rho)\}$ there exists a sequence of eigenfunctions $\{u_k(\rho)\} \in W_0^{1,2}(\Omega)$ satisfying $\int_\Omega \rho u_i(\rho) u_j(\rho)\, dx = \delta_{ij}$, the Kronecker delta, and forming a basis for $L^2(\Omega)$. When it is clear which ρ the $u_k(\rho)$ depend on, as in the previous orthogonality relationship, we shall simply write u_k. Denoting the $L^2(\Omega)$ inner product $\int_\Omega fg\, dx$ by $\langle f, g\rangle$ we employ the equivalent $L^2(\Omega)$ norms $\|u\|_2 = \langle u, u\rangle^{1/2}$ and $\|u\|_\rho = \langle \rho u, u\rangle^{1/2}$ for $\rho \in ad^*$. We denote by $E_k(\rho)$ the span of the first k eigenfunctions associated with ρ. Defining the Rayleigh quotient $\mathcal{R} : ad^* \times \left(W_0^{1,2}(\Omega) \setminus \{0\}\right) \to \mathbf{R}$,

$$\mathcal{R}(\rho, u) = \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega \rho u^2\, dx},$$

and the orthogonal projection $P_k(\rho) : W_0^{1,2}(\Omega) \to E_k(\rho)$, $P_0(\rho) \equiv 0$, while for $k > 1$,

$$P_k(\rho) u = \sum_{i=1}^{k} \langle \rho u, u_i\rangle u_i, \qquad \text{we recall}$$

Rayleigh's Principle.

$$\lambda_k(\rho) = \min_{\substack{u \in W_0^{1,2} \\ P_{k-1}(\rho)u = 0}} \mathcal{R}(\rho, u) \quad \text{and} \quad \frac{1}{\lambda_k(\rho)} = \max_{\substack{u \in W_0^{1,2} \\ P_{k-1}(\rho)u = 0}} \mathcal{R}^{-1}(\rho, u),$$

where $\mathcal{R}^{-1}(\rho, u)$ denotes the reciprocal of $\mathcal{R}(\rho, u)$ and the extremum is attained at each kth eigenfunction.

Noting that this latter formulation states that $1/\lambda_1(\rho)$ is the maximum of a set of linear weak* continuous functions we find

Lemma 3.1. *The map $\rho \mapsto 1/\lambda_1(\rho)$ is convex and weak* lower semicontinuous on ad^*.*

An alternate formulation of $\lambda_k(\rho)$, which does not require knowledge of u_{k-1} to obtain u_k is the

Minimax Principle. *$\lambda_k(\rho) = \min_{V \in W_k} \max_{u \in V} \mathcal{R}(\rho, u)$, where W_k is the class of k dimensional subspaces of $W_0^{1,2}(\Omega)$, the minimax being attained at each kth eigenfunction.*

Denoting by $\lambda_k(\alpha)$ $(\lambda_k(\beta))$ the kth eigenvalue of the homogeneous membrane with density α (β) this principle immediately yields

Lemma 3.2. *If $\rho \in ad^*$ then $\lambda_k(\beta) \le \lambda_k(\rho) \le \lambda_k(\alpha)$ for each k.*

We shall also have cause to invoke

Auchmuty's Principle [2].

$$\frac{-1}{2\lambda_k(\rho)} = \inf_{u \in W_0^{1,2}} \mathcal{A}_k(\rho, u), \quad \mathcal{A}_k(\rho, u) = \tfrac{1}{2}\|\nabla u\|_2^2 - \|(I - P_{k-1}(\rho))u\|_\rho,$$

the minimum being attained at a kth eigenfunction, normalized according to $\|\nabla u_k\|_2^2 = \|u_k\|_\rho = \lambda_k^{-1}(\rho)$.

We collect a number of the properties enjoyed by u_k.

Lemma 3.3. *$u_k \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \cap W_{\mathrm{loc}}^{2,2}(\Omega) \cap C^1(\Omega)$ and satisfies (3.1) almost everywhere in Ω. If $\partial\Omega$ is C^2 then $u_k \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \cap C^1(\overline{\Omega})$.*

Proof: One finds a uniform bound on u_k in terms of its L^2 norm. As a result, $\Delta u_k \in L^p$ for every $p \geq 1$, hence $u_k \in W_{\mathrm{loc}}^{2,p}(\Omega)$, and for $p > N$, $u_k \in C^1(\Omega)$. For details see Gilbarg and Trudinger [12] 8.15, 9.15, 7.26. ∎

Lemma 3.4. *u_k is superharmonic on any open set where it is nonnegative.*

Proof: For u_k, ϕ nonnegative on the open set $\mathcal{O}$, $\phi \in C_0^\infty(\mathcal{O})$,

$$-\int_{\mathcal{O}} u_k \Delta\phi \, dx = \int_{\mathcal{O}} \nabla u_k \cdot \nabla\phi \, dx = \lambda \int_{\mathcal{O}} \rho u_k \phi \, dx \geq 0,$$

i.e., $-\Delta u_k$ is a positive distribution on $\mathcal{O}$, hence a positive Radon measure on $\mathcal{O}$. As a consequence, Treves [26] 30.6, u_k is equal almost everywhere on $\mathcal{O}$ to a superharmonic function v. The lemma now follows from the continuity of u_k. ∎

Concerning preimages, $u^{-1}\{t\} = \{x \in \Omega : u(x) = t\}$, we shall need

Lemma 3.5. *([12] 7.7) If $u \in W_{\mathrm{loc}}^{1,1}(\Omega)$ then $\nabla u = 0$ a.e. on each of u's preimages.*

Lemma 3.6.

(i) If $\partial\Omega$ is C^2 then $|u_k^{-1}\{t\}| = 0$ for each $t \neq 0$.

(ii) $u_1^{-1}\{0\}$ is empty and $\lambda_1(\rho)$ is simple.

Proof: (i) Assume there exists a $t \in \mathbf{R}$ and $\epsilon > 0$ for which $\left|u_k^{-1}\{t\}\right| = \epsilon$. Lemma 3.5 implies that the set $U = u_k^{-1}\{t\} \cap \nabla u_k^{-1}\{0\}$ has measure ϵ as well. $u_k \in W^{2,2}(\Omega)$ implies $\nabla u_k \in W^{1,2}(\Omega)^N$ and since $\nabla u_k = 0$ in U, N more applications of Lemma 3.5 yield $\Delta u_k = 0$ a.e. in U. However, when $t \neq 0$, one finds

$$0 = -\Delta u_k(x) \neq \lambda_k(\rho)\rho(x)u_k(x) = \lambda_k(\rho)\rho(x)t \quad \text{a.e. in } U.$$

As $|U| > 0$, this contradicts the fact that u_k satisfies (3.1) a.e. in Ω. (ii) Rayleigh's principle with Lemma 3.5 implies that $|u_1|$ is an eigenfunction for $\lambda_1(\rho)$. As $|u_1| \geq 0$ on Ω it must be superharmonic on Ω, so $|u_1|$ must vanish identically on a ball about each of its zeros. As Ω is connected we see that $u_1^{-1}\{0\} \neq \emptyset$ implies $u_1^{-1}\{0\} = \Omega$. As each element of $\lambda_1(\rho)$'s eigenspace has fixed sign in Ω, $\lambda_1(\rho)$ must be simple. ∎

Note that if one had $u|_{\partial\Omega} = 0$, one would not need the smoothness hypothesis on $\partial\Omega$ in (i). $u_k^{-1}\{0\}$ is typically called u_k's **nodal set**, while the interior of the closure of a component of $\Omega \setminus u_k^{-1}\{0\}$ is a **nodal domain** of u_k. Conventional wisdom has it that an eigenfunction's nodal domains partition the membrane into independent submembranes. We close this section with a precise formulation of this remark.

Lemma 3.7. *If $\partial\Omega$ is C^2 and $\mathcal{O}$ is a nodal domain of u_k then $u_k|_{\mathcal{O}}$ belongs to $W_0^{1,2}(\mathcal{O})$ and is a first eigenfunction on $\mathcal{O}$.*

Proof: $u_k \in W^{1,2}(\mathcal{O}) \cap C^0(\overline{\mathcal{O}})$ and $u_k|_{\partial\mathcal{O}} = 0$ together imply, see Brezis [6] IX.17, that $u_k \in W_0^{1,2}(\mathcal{O})$. The remainder is clear. ∎

4. CONTINUOUS DEPENDENCE

In the preceding section, with the exception of Lemma 3.1, we fixed a $\rho \in ad_\gamma^*(\Omega)$ and examined properties of the associated eigenvalues $\{\lambda_k(\rho)\}$ and eigenfunctions $\{u_k(\rho)\}$. It is in this section that we begin our study of the maps $\rho \mapsto \lambda_k(\rho)$ and $\rho \mapsto u_k(\rho)$. Our first task is to establish their continuity in the weak* topology on ad_γ^*. As this space is metrizable we need only examine the behavior of $\lambda_k(\rho^n)$ and $u_k(\rho^n)$ as $\rho^n \overset{*}{\rightharpoonup} \rho$ in ad_γ^*. Our approach is roughly that of Attouch [1] 3.71, itself a combination of the efforts of Boccardo and Marcellini [4] and Kesavan [16]. Noting that $\rho^n \overset{*}{\rightharpoonup} \rho$ implies $\mathcal{G}_{\rho^n} \to \mathcal{G}_\rho$ in the uniform topology of bounded linear operators on

$W_0^{1,2}(\Omega)$, we remark that our results also follow from the more general approach of Kato [14].

Recalling the ambiguity inherent in the ordering of the eigenfunctions in an eigenspace corresponding to an eigenvalue with multiplicity greater than one, it should not be surprising that we can not treat the continuity of individual eigenfunctions. To compensate for this lack of individual compactness we can however consider finite spans. To make this precise we employ the notion of Mosco convergence. To a sequence of closed convex subsets $\{K^n\}$ of a reflexive Banach space V we associate the four limits

$$\begin{aligned} s\text{−Li}\, K^n &\equiv \{v \in V : \exists \{v^n\} \text{ s.t. } v^n \in K^n \text{ and } v^n \to v\}, \\ s\text{−Ls}\, K^n &\equiv \{v \in V : \exists \{K^{n_m}\} \subseteq \{K^n\} \text{ and } v^m \in K^{n_m} \text{ s.t. } v^m \to v\}, \\ w_{\rm seq}\text{−Li}\, K^n &\equiv \{v \in V : \exists \{v^n\} \text{ s.t } \mathrm{v^n \in K^n \text{ and } v^n \rightharpoonup v}\}, \\ w_{\rm seq}\text{−Ls}\, K^n &\equiv \{v \in V : \exists \{K^{n_m}\} \subseteq \{K^n\} \text{ and } v^m \in K^{n_m} \text{ s.t. } v^m \rightharpoonup v\}, \end{aligned}$$

where s and $\to$ refer to the strong(norm) topology on V while $w_{\rm seq}$ and $\rightharpoonup$ indicate V endowed with its weak topology. The seq subscript is meant to distinguish $w_{\rm seq}$−Li K^n and $w_{\rm seq}$−Ls K^n from the weak lower and upper Kuratowski limits which, by virtue of the fact that the weak topology on V is not first countable, do not yield to a sequential characterization. Noting that, $s\text{−Li}\, K^n \subseteq w_{\rm seq}\text{−Li}\, K^n$, $s\text{−Ls}\, K^n \subseteq w_{\rm seq}\text{−Ls}\, K^n$, we use

Definition 4.1. *A sequence of closed convex sets $\{K^n\}$ in a reflexive Banach space V is said to converge in the sense of Mosco to the set K when*

$$K = s\text{−Li}\, K^n = w_{\rm seq}\text{−Ls}\, K^n.$$

Before proceeding to this section's main result we dispense with,

Lemma 4.2. *If $f^n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$ while both $g^n \to g$ and $h^n \to h$ in $L^2(\Omega)$ then $\langle f^n g^n, h^n\rangle \to \langle fg, h\rangle$.*

Proof: $|\langle f^n g^n, h^n\rangle - \langle fg, h\rangle| \le \|f^n g^n\|_2 \|h^n - h\|_2 + |\langle f^n g^n - fg, h\rangle|$. The lemma follows on observing that $f^n g^n \rightharpoonup fg$ in $L^2(\Omega)$. ∎

For $\{\rho^n\} \subset ad_\gamma^*$ we established the existence, for each n, of a sequence of eigenvalues $\{\lambda_k(\rho^n)\}_k$ and a corresponding sequence of eigenfunctions $\{u_k(\rho^n)\}_k$ satisfying $\langle \rho u_i(\rho^n), u_j(\rho^n)\rangle = \delta_{ij}$ and $\|\nabla u_k(\rho^n)\|_2^2 = \lambda_k(\rho^n)$. We find it convenient to shorten $\lambda_k(\rho^n)$ to λ_k^n and $u_k(\rho^n)$ to u_k^n.

Proposition 4.3. *If $\rho^n \overset{*}{\rightharpoonup} \rho$ in ad_γ^* then*

(i) $\lambda_k^n \to \lambda_k(\rho)$ *for each* k.

(ii) If $\lambda_{k-1}(\rho) < \lambda_k(\rho) = \cdots = \lambda_{k+\nu_k-1}(\rho) < \lambda_{k+\nu_k}(\rho)$ *then* $\mathcal{S}_k^n \equiv \mathrm{span}\{u_i^n : i = k, \ldots, k+\nu_k-1\}$ *converges in the sense of Mosco in* $W_0^{1,2}(\Omega)$ *to* $\mathcal{S}_k(\rho) \equiv \mathrm{span}\{u_i(\rho) : i = k, \ldots, k+\nu_k-1\}$.

(iii) If $\lambda_k(\rho)$ *is simple then, for sufficiently large* n, *so is* λ_k^n. *Moreover,* $u_k^n \to u_k(\rho)$ *in* $W_0^{1,2}(\Omega)$.

Proof: We extract an arbitrary subsequence from $\{\rho^n\}$, which we continue to index by n. Lemma 3.2 implies that $\lambda_k(\beta) \le \lambda_k^n \le \lambda_k(\alpha)$, and hence $\|\nabla u_k^n\|_2^2 \le \lambda_k(\alpha)$. As a result we may (without relabeling) extract a further subsequence such that $\{\lambda_1^n\}$ converges in $\mathbf{R}$ and $\{u_1^n\}$ converges weakly in $W_0^{1,2}(\Omega)$, hence strongly in $L^2(\Omega)$. Continuing, we extract a still further subsequence in order that both $\{\lambda_2^n\}$ and $\{u_2^n\}$ converge in a similar fashion. Proceeding in this manner, denoting the limits by $\overline{\lambda}_k$ and $\overline{u}_k$, and finally indexing the standard diagonal subsequence by n, we find that for each k

$$\lambda_k^n \to \overline{\lambda}_k \qquad \text{and} \qquad u_k^n \rightharpoonup \overline{u}_k \quad \text{in } W_0^{1,2}(\Omega). \tag{4.1}$$

Also, $0 < \lambda_1^n < \lambda_2^n \le \cdots \uparrow \infty$ for each n, implies $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \cdots \uparrow \infty$. To see that $\{\overline{\lambda}_k\} \subseteq \{\lambda_k(\rho)\}$ we deduce from (4.1) and Lemma 4.2 that

$$\int_\Omega \nabla u_k^n \cdot \nabla v - \lambda_k^n \rho u_k^n v \, dx \to \int_\Omega \nabla \overline{u}_k \cdot \nabla v - \overline{\lambda}_k \rho \overline{u}_k v \, dx$$

for every $v \in W_0^{1,2}(\Omega)$. Consequently,

$$\int_\Omega \nabla \overline{u}_k \cdot \nabla v \, dx = \overline{\lambda}_k \int_\Omega \rho \overline{u}_k v \, dx \qquad \forall\, v \in W_0^{1,2}(\Omega). \tag{4.2}$$

(4.1) and Lemma 4.2 also combine to produce

$$\langle \rho u_i^n, u_j^n \rangle \to \langle \rho \overline{u}_i, \overline{u}_j \rangle = \delta_{ij}. \tag{4.3}$$

(4.2) and (4.3) now imply $\{\overline{\lambda}_k\} \subseteq \{\lambda_k(\rho)\}$. Note also that since $\lambda_k^n \to \overline{\lambda}_k$, $\lambda_k^n = \|\nabla u_k^n\|_2^2$, and $\overline{\lambda}_k = \|\nabla \overline{u}_k\|_2^2$ we actually have $\|\nabla u_k^n\|_2 \to \|\nabla \overline{u}_k\|_2$, which, in light of $u_k^n \rightharpoonup \overline{u}_k$ in $W_0^{1,2}(\Omega)$, implies that $u_k^n \to \overline{u}_k$ in $W_0^{1,2}(\Omega)$.

To establish $\{\lambda_k(\rho)\} \subseteq \{\overline{\lambda}_k\}$ we assume the existence of a $\overline{\lambda} \in \{\lambda_k(\rho)\} \setminus \{\overline{\lambda}_k\}$. Denoting by $\overline{u}$ a nontrivial element of $\overline{\lambda}$'s eigenspace we find $\langle \rho \overline{u}, \overline{u}_k \rangle = 0$ for every k. Without loss we assume further that $\|\overline{u}\|_\rho^{-1} = \overline{\lambda}$. Auchmuty's principle informs us that

$$\frac{-1}{2\lambda_k^n} \leq \mathcal{A}_k(\rho^n, \overline{u}). \tag{4.4}$$

Notice however, that

$$\begin{aligned} \mathcal{A}_k(\rho^n, \overline{u}) &= \tfrac{1}{2}\int_\Omega |\nabla \overline{u}|^2\,dx - \|(I - P_{k-1}(\rho^n))\overline{u}\|_{\rho^n} \\ &\to \tfrac{1}{2}\int_\Omega |\nabla \overline{u}|^2\,dx - \|\overline{u}\|_\rho = \frac{-1}{2\overline{\lambda}}, \end{aligned}$$

where the convergence is due to

$$P_{k-1}(\rho^n)\overline{u} = \sum_{j=1}^{k-1} \langle \rho^n \overline{u}, u_j^n \rangle u_j^n \to \sum_{j=1}^{k-1} \langle \rho \overline{u}, \overline{u}_j \rangle \overline{u}_j = 0 \quad \text{in } L^2(\Omega).$$

Letting n tend to infinity in (4.4) we find

$$\frac{-1}{2\overline{\lambda}_k} \leq \frac{-1}{2\overline{\lambda}}\ \forall\ k, \text{ that is } \overline{\lambda} \geq \overline{\lambda}_k\ \forall\ k,$$

a contradiction, as we have shown the $\overline{\lambda}_k$ to grow without bound. Hence $\{\overline{\lambda}_k\} = \{\lambda_k(\rho)\}$. As this result is independent of the original subsequence, (i) follows.

With respect to (ii) we show first that $w_{\mathrm{seq}}\text{–}\mathrm{Ls}\,\mathcal{S}_k^n \subseteq \mathcal{S}_k(\rho)$. Selecting a $u \in w_{\mathrm{seq}}\text{–}\mathrm{Ls}\,\mathcal{S}_k^n$ provides us with a sequence $\{u^m\}$, with $u^m \in \mathcal{S}_k^{n_m}$, i.e.,

$$u^m = \sum_{i=k}^{k+\nu_k-1} \langle \rho^{n_m} u^m, u_i^{n_m} \rangle u_i^{n_m}, \tag{4.5}$$

for which $u^m \rightharpoonup u$ in $W_0^{1,2}(\Omega)$. Recalling the diagonalization argument invoked above we find, without relabeling, a new subsequence for which $u_i^{n_m} \to u_i(\rho)$ in $W_0^{1,2}(\Omega)$ for each i. Passing to limit in (4.5) along this new subsequence produces

$$u = \sum_{i=k}^{k+\nu_k-1} \langle \rho u, u_i(\rho) \rangle u_i(\rho), \quad \text{i.e., } u \in \mathcal{S}_k(\rho).$$

Hence $w_{\mathrm{seq}}\text{–}\mathrm{Ls}\,\mathcal{S}_k^n \subseteq \mathcal{S}_k(\rho)$. In addition, since the convergence of each $u_i^{n_m}$ to $u_i(\rho)$ is strong, we have $s\text{–}\mathrm{Ls}\,\mathcal{S}_k^n = w_{\mathrm{seq}}\text{–}\mathrm{Ls}\,\mathcal{S}_k^n$.

As a consequence of Lemma 4.2 we saw above that weak $W_0^{1,2}(\Omega)$ convergence preserves the orthonormality of the u_i^n. Hence, $w_{\mathrm{seq}}\text{–}\mathrm{Ls}\,\mathcal{S}_k^n$ must contain a ν_k dimensional linear subspace of $W_0^{1,2}(\Omega)$. But $w_{\mathrm{seq}}\text{–}\mathrm{Ls}\,\mathcal{S}_k^n$ has already been shown to be contained in such a set, namely $\mathcal{S}_k(\rho)$, therefore $s\text{–}\mathrm{Ls}\,\mathcal{S}_k^n = w_{\mathrm{seq}}\text{–}\mathrm{Ls}\,\mathcal{S}_k^n = \mathcal{S}_k(\rho)$.

(ii) will follow upon demonstrating that $\mathcal{S}_k(\rho) \subseteq s\text{–}\mathrm{Li}\,\mathcal{S}_k^n$. If we assume that $v \in W_0^{1,2}(\Omega)$ is not an element of $s\text{–}\mathrm{Li}\,\mathcal{S}_k^n$, then there must exist a subsequence $\{\mathcal{S}_k^{n_m}\}$ such that $v \notin s\text{–}\mathrm{Ls}\,\mathcal{S}_k^{n_m}$. But arguing as above we know that $s\text{–}\mathrm{Ls}\,\mathcal{S}_k^{n_m} = \mathcal{S}_k(\rho)$, so $v \notin s\text{–}\mathrm{Li}\,\mathcal{S}_k^n \Rightarrow v \notin \mathcal{S}_k(\rho)$, i.e., $\mathcal{S}_k(\rho) \subseteq s\text{–}\mathrm{Li}\,\mathcal{S}_k^n$.

As $\lambda_k^n \to \lambda_k(\rho)$ for every k, there must exist an $M(k) > 0$ such that the multiplicity of λ_k^n is less than or equal to that of $\lambda_k(\rho)$ for $n \geq M(k)$. Consequently, if $\lambda_k(\rho)$ is simple, so is λ_k^n for n sufficiently large. In this case, the $\mathcal{S}_k^n$ and $\mathcal{S}_k(\rho)$ are each one-dimensional subspaces. Their Mosco convergence implies that upon choosing the u_k^n so that $\langle \rho u_k^n, u_k(\rho) \rangle$ eventually has fixed sign one finds $u_k^n \to u_k(\rho)$ in $W_0^{1,2}(\Omega)$. ∎

With respect to the argument offered by Attouch [1] 3.71, we remark that our use of Auchmuty's Principle greatly simplifies the proof of $\{\lambda_k(\rho\} \subseteq \{\overline{\lambda}_k\}$. In addition, under the general hypotheses of [1] 3.71 one may only prove the Mosco convergence of the $\mathcal{S}_k^n$ in $L^2(\Omega)$. In our case the convergence is in $W_0^{1,2}(\Omega)$.

Recalling the equivalence between the Mosco convergence of sets and the convergence of projections onto these sets, [1] 3.33, we deduce from the previous proposition that $P_{k-1}(\rho^n)u \to P_{k-1}(\rho)u$ in $L^2(\Omega)$ for each $u \in W_0^{1,2}(\Omega)$ when $\rho^n \overset{*}{\rightharpoonup} \rho$ in ad_γ^* and $\lambda_{k-1}(\rho) < \lambda_k(\rho)$. Recalling the dependence of the $\mathcal{A}_k$ on these projections we find, for example, that the maps $\rho \mapsto \mathcal{A}_1(\rho, u)$ and $\mathcal{A}_2(\rho, u)$ are weak* continuous on ad_γ^*.

As $\rho \mapsto \lambda_k(\rho)$ is a weak* continuous function on the weak* compact set $ad_\gamma^*(\Omega)$ we immediately find

Proposition 4.4. *For each $k = 1, \ldots,$ and γ between 0 and $|\Omega|$ there exist $\check{\rho}_k^\gamma$ and $\hat{\rho}_k^\gamma$ in ad_γ^* and corresponding eigenfunctions $\check{u}_k^\gamma$ and $\hat{u}_k^\gamma$ in $W_0^{1,2}(\Omega)$ such that*

$$\check{\lambda}_k^\gamma \equiv \lambda_k(\check{\rho}_k^\gamma) = \inf_{\rho \in ad_\gamma^*} \lambda_k(\rho), \qquad \frac{-1}{2\check{\lambda}_k^\gamma} = \mathcal{A}_k(\check{\rho}_k^\gamma, \check{u}_k^\gamma), \quad \textit{and}$$
$$\hat{\lambda}_k^\gamma \equiv \lambda_k(\hat{\rho}_k^\gamma) = \sup_{\rho \in ad_\gamma^*} \lambda_k(\rho), \qquad \frac{-1}{2\hat{\lambda}_k^\gamma} = \mathcal{A}_k(\hat{\rho}_k^\gamma, \hat{u}_k^\gamma).$$

5. STABILITY PROPERTIES

We now address the sensitivity of the optimal designs, $\check{\rho}_k^\gamma$ and $\hat{\rho}_k^\gamma$, and their associated eigenvalues, $\check{\lambda}_k^\gamma$ and $\hat{\lambda}_k^\gamma$, to changes in γ, the amount of material of density α. This section is dramatically different than its successors in the sense that the tools brought to bear on the minimizers, $\check{\rho}_k^\gamma$, prove equally effective on the maximizers, $\hat{\rho}_k^\gamma$. As such we shall devote ourselves to the study of $\gamma \mapsto \check{\rho}_k^\gamma$, indicating only briefly how corresponding results may be obtained for $\gamma \mapsto \hat{\rho}_k^\gamma$.

In the introduction, where $\Omega = (0,1)$ and we gave explicit formulae for the $\check{\rho}_k^\gamma$ and $\hat{\rho}_k^\gamma$, it was clear that $\gamma \mapsto \check{\rho}_k^\gamma$ and $\gamma \mapsto \hat{\rho}_k^\gamma$ were continuous maps form $\mathbf{R}$ to $ad^*(0,1)$ with the weak* topology inherited from $L^\infty(0,1)$. That is, the optimal designs were stable with respect to perturbations in the volume fractions of the two materials. To see how such a result can hold in higher dimensions, where we lack such detailed knowledge of the optimal designs, we must take advantage of the known dependence of ad_γ^* on γ. To this end we consider the indicator function $\mathcal{I}^\gamma : L^\infty(\Omega) \to \overline{\mathbf{R}}$,

$$\mathcal{I}^\gamma(\rho) = \begin{cases} 0, & \text{if } \rho \in ad_\gamma^*; \\ \infty, & \text{otherwise.} \end{cases}$$

With this we may reformulate the minimum problem to read

$$\lambda_k(\check{\rho}_k^\gamma) = \inf_{\rho \in ad^*} (\lambda_k(\rho) + \mathcal{I}^\gamma(\rho)).$$

Now, when $\gamma_n \to \gamma$ we arrive at a sequence of such minimization problems and a sequence $\{\check{\rho}_k^{\gamma_n}\} \subset ad^*$ of corresponding minimizers. The weak* sequential compactness of ad^* implies the existence of a subsequence $\{\check{\rho}_k^{\gamma_{n_j}}\} \subset \{\check{\rho}_k^{\gamma_n}\}$ and a function $\overline{\rho} \in ad^*$ for which $\check{\rho}_k^{\gamma_{n_j}} \xrightarrow{*} \overline{\rho}$. As $\check{\rho}_k^{\gamma_{n_j}} \in ad_{\gamma_{n_j}}^*$ for each n_j we see from Proposition 2.8 that $\overline{\rho} \in ad_\gamma^*$. The question that remains of course, is whether or not $\overline{\rho}$ actually minimizes λ_k on ad_γ^*. This sort of question, whether the solutions to a sequence of variational problems cluster at a solution to some limiting variational problem, finds its answer in De Giorgi's notion of Γ convergence. As our variational problems are ones of minimization we follow Attouch [1] in using the more descriptive terminology of epi convergence.

Definition 5.1. *([1] 1.14) Let (V, τ) be a metrizable space and $\{F^n\}$ a sequence of functions from V into $\overline{\mathbf{R}}$. $\{F^n\}$ is said to $\tau-$epi converge to F when*

(i) $\forall\, v \in V\ \exists \{v^n\} \subset V$ *for which* $v^n \xrightarrow{\tau} v$ *and* $F^n(v^n) \to F(v)$.

(ii) And $\forall\, v \in V$ *if* $\{v^n\} \subset V$ *and* $v^n \xrightarrow{\tau} v$ *then* $F(v) \le \liminf_{n\to\infty} F^n(v^n)$.

When F is the $\tau-$epi limit of some $\{F^n\}$ we write $F = \tau{-}\lim_{\mathrm{e}} F^n$. Recalling that the epigraph of $F : V \to \overline{\mathbf{R}}$ is the set

$$\text{epi } F = \{(v,t) \in V \times \overline{\mathbf{R}} : F(v) \le t\},$$

we find justification for the epi terminology in the fact that the epigraph of the epi limit of a sequence of functions is the Kuratowski limit of the corresponding sequence of epigraphs, [1] 1.39.

The utility of epi convergence lies in its status as a **variational** convergence, i.e., in its ability to (i) allow one to go to the limit in a sequence of minimization problems and (ii) withstand continuous perturbations. As each of these properties shall prove useful in this section, as well as in Part II, we take the time to present them in a precise manner.

Proposition 5.2. *([1] 1.10) If* $F = \tau{-}\lim_{\mathrm{e}} F^n$ *on* (V,τ) *and* $\{v^n\} \subset V$ *satisfies* $F^n(v^n) \le \inf\{F^n(v) + \epsilon^n;\ v \in V\}$, *where* $\epsilon^n \to 0$, *then for every* $\tau-$*converging subsequence* $v^{n_j} \xrightarrow{\tau} \overline{v}$ *we have*

$$F^{n_j}(v^{n_j}) \to F(\overline{v}) = \inf_{v \in V} F(v).$$

Proposition 5.3. *([1] 2.15) If* $F = \tau{-}\lim_{\mathrm{e}} F^n$ *on* (V,τ) *and* $G : V \to \overline{\mathbf{R}}$ *is a* τ *continuous function, then* $\tau{-}\lim_{\mathrm{e}}(F^n + G) = \tau{-}\lim_{\mathrm{e}} F^n + G$.

Proposition 5.2 states that $\overline{\rho}$ will indeed be a minimizer of $\lambda_k + \mathcal{I}^\gamma$, hence deserving of the title $\check{\rho}_k^\gamma$, if in fact $\lambda_k + \mathcal{I}^\gamma = w^*{-}\lim_{\mathrm{e}}(\lambda_k + \mathcal{I}^{\gamma_n})$. As we have already demonstrated the weak* continuity of $\rho \mapsto \lambda_k(\rho)$, this will follow if we can only show that $\mathcal{I}^\gamma = w^*{-}\lim_{\mathrm{e}} \mathcal{I}^{\gamma_n}$. This will follow from yet another intimate connection between set and functional convergence.

Proposition 5.4. *([1] 1.40) Let $\{K^n\}$ be a sequence of subsets of a topological space (V, τ) and denote by $\{I_{K^n}\}$ the corresponding sequence of indicator functions. Then $K = \tau{-}\mathrm{Lim}\, K^n$ if and only if $I_K = \tau{-}\mathrm{lim_e}\, I_{K^n}$.*

We now have all the pieces to the puzzle. For Propositions 5.4 and 2.8 yield $\mathcal{I}^\gamma = w^*{-}\mathrm{lim_e}\, \mathcal{I}^{\gamma_n}$. Propositions 5.3 and 4.3 combine to bring us

$$\lambda_k + \mathcal{I}^\gamma = w^*{-}\mathrm{lim_e}\, (\lambda_k + \mathcal{I}^{\gamma_n}).$$

Citing Proposition 5.2 now yields the desired result.

Proposition 5.5. *If $\gamma_n \to \gamma$ then $\check{\lambda}_k^{\gamma_n} \to \check{\lambda}_k^\gamma$ and there exists a subsequence $\{\gamma_{n_j}\} \subset \{\gamma_n\}$ such that $\check{\rho}_k^{\gamma_{n_j}} \stackrel{*}{\rightharpoonup} \check{\rho}_k^\gamma$ in ad^*. If $\check{\rho}_k^\gamma$ is unique then one need not pass to a subsequence.*

Recalling Lemma 3.2, i.e., $\lambda_k(\beta) \le \lambda_k(\rho) \le \lambda_k(\alpha)$, we may expect that increasing the amount of α material present in a membrane can only increase the associated natural frequencies.

Proposition 5.6. *(i) $\gamma \mapsto \check{\lambda}_k^\gamma$ is nondecreasing. (ii) $\gamma \mapsto \check{\lambda}_1^\gamma$ is strictly increasing.*

Proof: (i) If $\gamma_1 \le \gamma_2$ then there exists a $\psi \in ad^*_{\gamma_1}$ such that $\check{\rho}_k^{\gamma_2}(x) \le \psi(x)$ a.e. in Ω. Consequently, $\mathcal{R}(\psi, u) \le \mathcal{R}(\check{\rho}_k^{\gamma_2}, u)$ for every $u \in W_0^{1,2}(\Omega)$. The Minimax Principle now yields

$$\check{\lambda}_k^{\gamma_1} \le \lambda_k(\psi) \le \lambda_k(\check{\rho}_k^{\gamma_2}) = \check{\lambda}_k^{\gamma_2}.$$

(ii) If $\gamma_1 < \gamma_2$ then there exists a $\psi \in ad^*_{\gamma_1}$ such that $\check{\rho}^{\gamma_2}(x) \le \psi(x)$ a.e. in Ω, with strict inequality on a set of positive measure. Rayleigh's Principle, in conjunction with the fact that $\check{u}^{\gamma_2}$ does not vanish on Ω, now yields

$$\check{\lambda}_1^{\gamma_1} \le \lambda_1(\psi) \le \mathcal{R}(\psi, \check{u}^{\gamma_2}) < \mathcal{R}(\check{\rho}^{\gamma_2}, \check{u}^{\gamma_2}) = \check{\lambda}_1^{\gamma_2}. \blacksquare$$

With respect to the map $\gamma \mapsto \hat{\rho}_k^\gamma$, we reformulate the maximum problem to read

$$\lambda_k(\hat{\rho}_k^\gamma) = \sup_{\rho \in ad^*} (\lambda_k(\rho) - \mathcal{I}^\gamma(\rho)).$$

The relevant notion here is hypo convergence, after

$$\text{hypo } F = \{(v,t) \in V \times \overline{\mathbf{R}} : F(v) \geq t\}.$$

That no new apparatus need be introduced is apparent from the fact that $F = \tau\text{–}\lim_{\mathrm{h}} F^n$, i.e., F is the τ-hypo limit of $\{F^n\}$, if and only if $-F = \tau\text{–}\lim_{\mathrm{e}}(-F^n)$. Hence, $\lambda_k - \mathcal{I}^\gamma = w^*\text{–}\lim_{\mathrm{h}}(\lambda_k - \mathcal{I}^{\gamma_n})$. The hypo version of Proposition 5.2, [1] 1.44, then yields,

Proposition 5.7. *If $\gamma_n \to \gamma$ then $\hat{\lambda}_k^{\gamma_n} \to \hat{\lambda}_k^\gamma$ and there exists a subsequence $\{\gamma_{n_j}\} \subset \{\gamma_n\}$ such that $\hat{\rho}_k^{\gamma_{n_j}} \overset{*}{\rightharpoonup} \hat{\rho}_k^\gamma$ in ad^*. If $\hat{\rho}_k^\gamma$ is unique then $\hat{\rho}_k^{\gamma_n} \overset{*}{\rightharpoonup} \hat{\rho}_k^\gamma$ in ad^*.*

In a similar fashion we may reverse the argument in Proposition 5.6 and arrive at identical monotonicity results for $\gamma \mapsto \hat{\lambda}_k^\gamma$.

Proposition 5.8. *(i) $\gamma \mapsto \hat{\lambda}_k^\gamma$ is nondecreasing. (ii) $\gamma \mapsto \hat{\lambda}_1^\gamma$ is strictly increasing.*

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REFERENCES

[1] Attouch, H., *Variational Convergence for Functions and Operators*, Pitman, Boston, 1984.

[2] Auchmuty, G., *Dual variational principles for eigenvalue problems*, in Nonlinear Functional Analysis and its Applications, F. Browder, ed., AMS, 1986, pp. 55–71.

[3] Barbu, V. and Precupanu, T., *Convexity and Optimization in Banach Spaces*, Reidel, Boston, 1986.

[4] Boccardo, L. and Marcellini, P., *Sulla convergenza delle soluzioni di disequazioni variazionali*, Ann. Mat. Pura Appl. **4**(110), 1977, pp. 137–159.

[5] Bourbaki, N., *Topological Vector Spaces, Chapters 1–5*, Springer Verlag, New York, 1987.

[6] Brezis, H., *Analyse Fonctionelle*, Masson, Paris, 1983.

[7] Cea, J. and Malanowski, K., *An example of a max-min problem in partial differential equations*, SIAM J. Control, **8**(3), 1970, pp. 305–316.

[8] Cosner C. and Schmitt, K., *On the geometry of the level sets of positive solutions of semilinear elliptic equations*, Rocky Mtn. J. Math. **18**(2), 1988, pp. 277–286.

[9] Ekeland, I. and Temam, R., *Convex Analysis and Variational Problems*, North Holland, Amsterdam, 1976.

[10] Friedland, S., *Extremal eigenvalue problems defined for certain classes of functions*, Arch. Rat. Mech. Anal., **67**(1), 1977, pp. 73–81.

[11] Gidas, B., Ni, W., Nirenberg, L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68**, 1979, pp. 209–243.

[12] Gilbarg, D. and Trudinger, N., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1983.

[13] Jouron, C., *Sur un problème d'optimisation ou la contrainte porte sur la fréquence fondamentale*, R.A.I.R.O. Analyse numérique **12**(4), 1978, pp. 349–374.

[14] Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1984.

[15] Kawohl, B., *Rearrangements and Convexity of Level Sets in PDE*, Springer Verlag, New York, 1985.

[16] Kesavan, S., *Homogenization of elliptic eigenvalue problems* I, Appl. Math. Optm. **5**, 1979, pp. 153–167.

[17] Knaster, B., Kuratowski, C., and Mazurkiewicz, S., *Eine beweis des fixpunktsatzes für n-dimensionale simplexe*, Fund. Math. **14**, 1929, pp. 132–137.

[18] Kohn, R. and Strang, G., *Optimal design and relaxation of variational problems I, II, III*, Comm. Pure Appl. Math. **39**, 1986, pp. 113–137, 139–182, and 353–377.

[19] Krein, M.G., *On certain problems on the maximum and minimum of characteristic values and on the Lyapunov zones of stability*, AMS Translations Ser. **2**(1), 1955, pp. 163–187.

[20] Mikhlin, S.G., *Variational Methods in Mathematical Physics*, Macmillan, New York, 1964.

[21] Olhoff, N., *Optimal design with respect to structural eigenvalues*, Proc. 15th Int. Cong. of Theoretical and Applied Mechanics, 1980, Rimrott and Tabarrok, eds., North-Holland, Amsterdam, 1980, pp. 133–149.

[22] Roberts, A. and Varberg, D., *Convex Functions*, Academic Press, New York, 1973.

[23] Stuart, C.A. and Toland, J.F., *A variational method for boundary value problems with discontinuous nonlinearities*, J. London Math. Soc. **2**(21), 1980, pp. 319–328.

[24] Tahraoui, R., *Quelques remarques sur le controle des valeurs propres*, in Nonlinear PDE's and their Applications: College de France Seminar, Volume VIII, H. Brezis and J.L. Lions, eds., Longman, Essex, 1988,

pp. 176–213.

[25] Tartar, L., *Compensated compactness and applications to partial differential equations*, in Nonlinear Analysis and Mechanics: Heriot-Watt Symposium IV, R. J. Knops, ed., Pitman, London, 1979.

[26] Treves, F., *Basic Linear Partial Differential Equations*, Academic Press, New York, 1975.