EXTREMAL EIGENVALUE PROBLEMS FOR COMPOSITE MEMBRANES, II

Steven J. Cox† and Joyce R. McLaughlin‡

† Courant Institute of Mathematical Sciences,
251 Mercer Street, New York, NY 10012

‡ Department of Mathematical Sciences,
Rensselaer Polytechnic Institute, Troy, NY 12180

Abstract: Given an open bounded connected set $\Omega \subset \mathbf{R}^N$ and a prescribed amount of two homogeneous materials of different density, for small k we characterize the distribution of the two materials in Ω that extremizes the kth eigenvalue of the resulting clamped membrane. We show that these extremizers vary continuously with the proportion of the two constituents. The characterization of the extremizers in terms of level sets of associated eigenfunctions provides geometric information on their respective interfaces. Each of these results generalizes to N dimensions the now classical one dimensional work of M.G. Krein.

In Part I we considered those composite membranes on $\Omega \subset \mathbf{R}^N$ with density in

$$ad_\gamma = \{\alpha\chi + \beta(1-\chi) : \chi \text{ is the characteristic function}$$
$$\text{of a subset of } \Omega \text{ of measure } \gamma\},$$

for fixed $0 < \alpha < \beta$ and $0 < \gamma < |\Omega|$. Denoting by $\lambda_k(\rho)$ the kth eigenvalue and by $u_k(\rho)$ a kth eigenfunction of the clamped membrane on Ω with density ρ we established the existence, and continuous dependence in γ, of pairs $(\check{\rho}_k^\gamma, \check{u}_k^\gamma)$ and $(\hat{\rho}_k^\gamma, \hat{u}_k^\gamma)$ in $ad_\gamma^* \times W_0^{1,2}(\Omega)$ for which

$$\check{\lambda}_k^\gamma \equiv \lambda_k(\check{\rho}_k^\gamma) = \inf_{\rho \in ad_\gamma^*} \lambda_k(\rho), \qquad \frac{-1}{2\check{\lambda}_k^\gamma} = \mathcal{A}_k(\check{\rho}_k^\gamma, \check{u}_k^\gamma), \quad \text{and}$$
$$\hat{\lambda}_k^\gamma \equiv \lambda_k(\hat{\rho}_k^\gamma) = \sup_{\rho \in ad_\gamma^*} \lambda_k(\rho), \qquad \frac{-1}{2\hat{\lambda}_k^\gamma} = \mathcal{A}_k(\hat{\rho}_k^\gamma, \hat{u}_k^\gamma),$$

where $ad_\gamma^* = \{\rho \in L^\infty(\Omega) : \alpha \le \rho(x) \le \beta, \ \int_\Omega \rho\, dx = \alpha\gamma + \beta(|\Omega| - \gamma)\}$ is the closure of ad_γ with respect to the weak* topology on $L^\infty(\Omega)$ and $\mathcal{A}_k : L^\infty(\Omega) \times W_0^{1,2}(\Omega) \to \mathbf{R}$ is Auchmuty's functional

$$\mathcal{A}_k(\rho, u) = \tfrac{1}{2}\|\nabla u\|_2^2 - \|(I - P_{k-1}(\rho))u\|_\rho,$$

with $\|f\|_2$ denoting the $L^2(\Omega)$ norm $(\int_\Omega f^2\, dx)^{1/2} = \langle f, f\rangle^{1/2}$, $\|f\|_\rho$ the weighted norm $(\int_\Omega \rho f^2\, dx)^{1/2}$ and $P_{k-1}(\rho)u = \sum_{j=1}^{k-1} \langle \rho u_j(\rho), u\rangle u_j(\rho)$.

Motivated by the recent success in the analysis of composite materials, e.g., [18], and the simple one-dimensional constructions of Krein [19] discussed in Part I, we embark on a characterization of the optimal designs $\check{\rho}_k^\gamma$ and $\hat{\rho}_k^\gamma$.

Our first concern is with finding necessary conditions of optimality. As with many problems in the calculus of variations these can be obtained through either indirect or direct means. By indirect we refer to the condition that zero lie in the gradient of the Lagrangian assembled from the cost and constraint functionals. This strategy has been applied by Jouron

[13] and more recently Tahraoui [24] in contexts that include our membrane problem. In contrast, the direct approach simply amounts to swapping the order in which the various infima and suprema are taken when the cost functional is itself characterized by a variational principle. For example, in choosing the conductivity of a device to minimize average temperature one arrives at a minmax problem that is affine in conductivity and concave in temperature. That minmax = maxmin then follows by standard arguments, and one finds the desired optimality conditions. Two obvious advantages of such an approach over the indirect are its ability to encompass nonsmooth problems and its close tie to the geometry of the problem, often obscured by the more abstract Lagrange multiplier rule. To our knowledge, this direct approach has yet to be exploited in the context of an extremal eigenvalue problem.

In §6 we take the direct approach to characterizing $\check{\rho}_k^\gamma$. In this case, the optimality conditions are an immediate consequence of a minmin principle. In addition to [13] and [24] we recall that Friedland [10] has obtained partial results in this direction. Although not ostensibly interested in optimality criteria, in applying the Convoy Principle to finite-dimensional approximations of the resolvent, $\mathcal{G}_\rho$, he was able to prove

Friedland [10]. *$\check{\rho}_k^\gamma \in ad_\gamma$ for each k.*

Hence, as in the one-dimensional case, λ_k attains its minimum on ad_γ, the original set of admissible composites.

In §7 we take the direct approach to characterizing $\hat{\rho}_k^\gamma$. Here $\hat{\lambda}_k^\gamma$ is a maxmin and only for $k = 1$ are we able to show the equivalence to minmax and obtain the desired optimality criteria.

Finally, In §8 we exploit the optimality conditions obtained in §7 in describing the shape of the interface defined by $\hat{\rho}^\gamma$.

6. MINIMIZING λ_k

Proposition 4.4 delivers a $\check\rho_k^\gamma \in ad_\gamma^*$ and a $\check u_k^\gamma \in W_0^{1,2}(\Omega)$ for which

$$\frac{-1}{2\check\lambda_k^\gamma} = \mathcal{A}_k(\check\rho_k^\gamma, \check u_k^\gamma) = \inf_{\rho\in ad_\gamma^*} \inf_{u\in W_0^{1,2}} \mathcal{A}_k(\rho, u).$$

As the order in which the two infima are taken is immaterial, we arrive at the following optimality criteria for $\check\rho_k^\gamma$ and $\check u_k^\gamma$,

$$\mathcal{A}_k(\check\rho_k^\gamma, \check u_k^\gamma) \le \mathcal{A}_k(\check\rho_k^\gamma, u) \qquad \forall\, u \in W_0^{1,2}(\Omega), \tag{6.1}$$

$$\mathcal{A}_k(\check\rho_k^\gamma, \check u_k^\gamma) \le \mathcal{A}_k(\rho, \check u_k^\gamma) \qquad \forall\, \rho \in ad_\gamma^*. \tag{6.2}$$

The first condition merely states that $\check u_k^\gamma$ is a kth eigenfunction for $\check\rho_k^\gamma$, satisfying $\|\nabla \check u_k^\gamma\|_2^2 = \|\check u_k^\gamma\|_{\check\rho_k^\gamma} = 1/\check\lambda_k^\gamma$. The second condition however, is much more interesting. On recalling the definition of $\mathcal{A}_k$, this becomes

$$\int_\Omega \rho(\check u_k^\gamma)^2\,dx - \int_\Omega \rho(P_{k-1}(\rho)\check u_k^\gamma)^2\,dx \le \int_\Omega \check\rho_k^\gamma(\check u_k^\gamma)^2\,dx \qquad \forall\,\rho\in ad_\gamma^*.$$

Initially we shall take $k = 1$, in which case we find

$$\int_\Omega \rho(\check u^\gamma)^2\,dx \le \int_\Omega \check\rho^\gamma(\check u^\gamma)^2\,dx \qquad \forall\,\rho\in ad_\gamma^*. \tag{6.3}$$

This maximum principle yields pointwise information in a number of ways. For example, see Cea and Malanowski [7] 3.2 for a nice geometric argument or Barbu and Precupanu [3] 3.1.7 for the standard Lagrange multiplier treatment. In either case, taking $\check u^\gamma(x) > 0$ in Ω, one finds

Proposition 6.1. *There exists an $\check\ell \ge 0$ such that for each $x \in \Omega$,*

$$\begin{aligned} \check\rho^\gamma(x) = \beta &\Rightarrow \check u^\gamma(x) \ge \check\ell,\\ \alpha < \check\rho^\gamma(x) < \beta &\Rightarrow \check u^\gamma(x) = \check\ell,\\ \check\rho^\gamma(x) = \alpha &\Rightarrow \check u^\gamma(x) \le \check\ell. \end{aligned} \tag{6.4}$$

Note that, unless $\gamma = |\Omega|$, we may assume $\check\ell > 0$. Now (6.1) implies that $\check\rho^\gamma$ determines $\check u^\gamma$ up to a change in sign. Conversely,

Corollary 6.2. *(i) $\breve{\rho}^\gamma$ may be chosen from ad_γ. (ii) If $\partial\Omega$ is C^2 then $\breve{u}^\gamma$ uniquely determines $\breve{\rho}^\gamma$.*

Proof: (i): (6.3) informs us that $\breve{\rho}^\gamma$ maximizes a continuous linear functional over a convex compact set. Hence, see e.g., Bourbaki [5] II.54, $\breve{\rho}^\gamma$ may be chosen an extreme point of ad^*_γ, i.e., an element of ad_γ. With regard to (ii) we observe in (6.4) that the set on which $\breve{\rho}^\gamma$ is neither α or β is precisely the preimage $(\breve{u}^\gamma)^{-1}\{\breve{\ell}\}$. With the C^2 hypothesis on $\partial\Omega$ we can invoke Lemma 3.6(i) and conclude that $|(\breve{u}^\gamma)^{-1}\{\breve{\ell}\}| = 0$, i.e., $\breve{\rho}^\gamma \in ad_\gamma$. Consequently $\breve{u}^\gamma$, through (6.4), uniquely determines $\breve{\rho}^\gamma$. ∎

Remark 6.3. *With $\breve{\rho}^\gamma \in ad_\gamma$ one may deduce from Proposition 5.5 that when $\gamma_n \to \gamma$ a subsequence may be found for which $\breve{\rho}^{\gamma_{n_j}}(x) \to \breve{\rho}^\gamma(x)$ for almost every $x \in \Omega$.*

Krein, upon characterizing $\breve{\rho}^\gamma$, was able to exploit the nodal structure of one dimensional eigenfunctions and arrive at a full characterization of $\breve{\rho}^\gamma_k$ for all $k > 1$. In the remainder of this section we investigate the degree to which Krein's arguments may be pushed to higher dimensions.

As $k > 1$ we can not assume $\breve{\lambda}^\gamma_k$ to be simple. Consequently, we focus our attention on $\lambda_k(\breve{\rho}^\gamma_k)$'s eigenspace, $\mathcal{S}_k(\breve{\rho}^\gamma_k)$. When Ω_i is a nodal domain for some element of $\mathcal{S}_k(\breve{\rho}^\gamma_k)$ we shall denote by $\sigma^i_1(\rho)$ the first eigenvalue of

$$-\Delta w = \sigma\rho w \quad \text{in } \Omega_i, \quad w|_{\partial\Omega_i} = 0, \quad \rho \in ad^*(\Omega_i).$$

Proposition 6.4. *If $\partial\Omega$ is C^2 and there exists a $\mathbf{u} \in \mathcal{S}_k(\breve{\rho}^\gamma_k)$ with k nodal domains, $\{\Omega_i\}_{i=1}^k$, then there exists a set, $\{\breve{\ell}_i\}_{i=1}^k$, $\breve{\ell}_i \geq 0$, such that for each $x \in \Omega_i$,*

$$\breve{\rho}^\gamma_k(x) = \begin{cases} \beta, & \text{if } |\mathbf{u}(x)| > \breve{\ell}_i; \\ \alpha, & \text{if } |\mathbf{u}(x)| \leq \breve{\ell}_i. \end{cases} \tag{6.5}$$

Proof: To see where the hypothesis of k nodal domains arises we begin by assuming that $\mathbf{u}$ has the n_k nodal domains, Ω_i. As a consequence of Lemma

3.7 we see that $\mathbf{u}|_{\Omega_i} \in W_0^{1,2}(\Omega_i)$ and that $\check{\lambda}_k^\gamma = \sigma_1^i(\check{\rho}_k^\gamma|_{\Omega_i})$, i.e., $\check{\lambda}_k^\gamma$ is the first eigenvalue of Ω_i with density $\check{\rho}_k^\gamma|_{\Omega_i}$. We now solve

$$\int_{\Omega_i} \check{\rho}_k^\gamma(x)\,dx = \alpha\gamma_i + \beta(|\Omega_i| - \gamma_i)$$

for γ_i, $i = 1, \ldots, n_k$ and compute

$$\check{\sigma}_1^{\gamma_i} = \inf\{\sigma_1^i(\rho) : \rho \in ad^*_{\gamma_i}(\Omega_i)\} \text{ and } \hat{\sigma}_1^{\gamma_i} = \sup\{\sigma_1^i(\rho) : \rho \in ad^*_{\gamma_i}(\Omega_i)\},$$

As a result, $\check{\lambda}_k^\gamma$ must dominate each of the $\check{\sigma}_1^{\gamma_i}$, which, without loss, we assume to be ordered with increasing i, i.e.,

$$\check{\sigma}_1^{\gamma_1} \le \check{\sigma}_1^{\gamma_2} \le \cdots \le \check{\sigma}_1^{\gamma_{n_k}} \le \check{\lambda}_k^\gamma. \tag{6.6}$$

If equality holds throughout (6.6) then $\check{\rho}_k^\gamma|_{\Omega_i}$ minimizes σ_1^i over $ad^*_{\gamma_i}(\Omega_i)$ and (6.5) follows from our previous optimality conditions (6.4). If inequality holds in (6.6) then there must exist an m such that

$$\check{\sigma}_1^{\gamma_1} \le \cdots \le \check{\sigma}_1^{\gamma_m} < \check{\sigma}_1^{\gamma_{m+1}} = \cdots = \check{\sigma}_1^{\gamma_{n_k}} = \check{\lambda}_k^\gamma. \tag{6.7}$$

We'll now show that (6.7) gives rise to a $\psi \in ad^*_\gamma$ for which $\lambda_k(\psi) < \check{\lambda}_k^\gamma$.

(a) If $m = n_k$ set $\zeta_i = \gamma_i$, $i = 1, \ldots, n_k$ and proceed to (c).

(b) If $m \ne n_k$ then $\check{\sigma}_1^{\gamma_j} = \check{\lambda}_k^\gamma < \hat{\sigma}_1^{\gamma_j}$, $j = m+1, \ldots, n_k$ implies via the the fact that $\gamma \mapsto \hat{\sigma}_1^\gamma$ is continuous and increasing, Proposition 5.6(ii), the existence of $\delta_i > 0$ such that

$$\check{\lambda}_k^\gamma < \hat{\sigma}_1^{\gamma_{m+i} - \delta_i}, \quad i = 1, \ldots, n_k - m.$$

In addition, there exist $\epsilon_i > 0$ such that

$$\check{\sigma}_1^{\gamma_i + \epsilon_i} < \check{\lambda}_k^\gamma, \quad i = 1, \ldots, m.$$

Now choose $\eta_i \in]0, \delta_i[$ so small that if $\eta = \sum_{i=1}^{n_k - m} \eta_i$ then $\dfrac{\eta}{m} < \min_{1 \le i \le m} \epsilon_i$. Setting

$$\zeta_i = \begin{cases} \gamma_i + \frac{\eta}{m}, & \text{if } i = 1, \ldots, m; \\ \gamma_i - \eta_{i-m}, & \text{if } i = m+1, \ldots, n_k, \end{cases}$$

in effect extracts a small amount of material with density α from the Ω_i, $i \geq m+1$, hence decreasing their fundamental tones, and adds it to the Ω_j, $j \leq m$, thereby increasing their fundamental tones.

(c) By construction we now have $\gamma = \sum_{i=1}^{n_k} \zeta_i = \sum_{i=1}^{n_k} \gamma_i$ and

$$\check{\sigma}_1^{\zeta_i} < \check{\lambda}_k^{\gamma} \leq \hat{\sigma}_1^{\zeta_i}, \quad i = 1, \ldots, n_k, \tag{6.8}$$

where $\check{\sigma}_1^{\zeta_i}$ ($\hat{\sigma}_1^{\zeta_i}$) is the minimum (maximum) of $\sigma_1^i(\rho)$ on Ω_i as ρ varies over $ad^*_{\zeta_i}(\Omega_i)$. We now select a $\mu \in]\max_{1 \leq i \leq n_k} \check{\sigma}_1^{\zeta_i}, \check{\lambda}_k^{\gamma}[$. As $ad^*_{\zeta_i}(\Omega_i)$ is convex, hence connected, and $\rho \mapsto \sigma_1^i(\rho)$ is continuous, (6.8) yields the existence of a $\rho^i \in ad^*_{\zeta_i}(\Omega_i)$ such that $\sigma_1^i(\rho^i) = \mu$ for each i. Setting

$$\psi(x) = \rho^i(x) \quad \text{when } x \in \Omega_i, \qquad \text{we find}$$

$$\int_\Omega \psi \, dx = \sum_{i=1}^{n_k} \int_{\Omega_i} \rho^i \, dx = \alpha\gamma + \beta(|\Omega| - \gamma),$$

i.e., $\psi \in ad^*_\gamma(\Omega)$. Denoting by $\{v_i(\psi)\}_{i=1}^{k-1}$ the first $k-1$ eigenfunctions corresponding to ψ on Ω we next construct a $\tilde{u} \in W_0^{1,2}(\Omega)$ such that $\mathcal{R}(\psi, \tilde{u}) = \mu$,

$$\int_\Omega \psi \tilde{u}^2 \, dx = 1, \quad \text{and} \quad \int_\Omega \psi \tilde{u} v_i(\psi) \, dx = 0 \quad i = 1, \ldots, k-1. \tag{6.9}$$

We begin with u_1^i, the first eigenfunction on Ω_i for ρ^i. Denoting by χ_i the characteristic function of Ω_i we combine the u_1^i according to

$$\tilde{u} = \sum_{i=1}^{n_k} b_i \chi_i u_1^i, \tag{6.10}$$

choosing the b_i, not all identically zero, so as to satisfy (6.9). Substituting (6.10) into (6.9) we see that for such a $b = (b_1, b_2, \ldots, b_{n_k})$ to exist it is both necessary and sufficient that the $(k-1) \times n_k$ matrix with ij^{th} element $\int_{\Omega_j} \rho^j u_1^j v_i(\psi) \, dx$ have rank less than n_k. This condition is clearly satisfied when $n_k = k$, i.e., when $\mathbf{u}$ possesses k nodal domains. Under this assumption, we compute

$$\int_\Omega |\nabla \tilde{u}|^2\, dx = \sum_{i=1}^k \int_\Omega |b_i \nabla u_1^i|^2\, dx = \mu \sum_{i=1}^k \int_{\Omega_i} \rho^i (b_i u_1^i)^2\, dx = \mu.$$

Observing that $\tilde{u}$ is orthogonal with weight ψ to the first $k-1$ eigenfunctions corresponding to ψ, we find via Rayleigh's principle

$$\lambda_k(\psi) \leq \frac{\int_\Omega |\nabla \tilde{u}|^2\, dx}{\int_\Omega \psi \tilde{u}^2\, dx} = \mu < \check{\lambda}_k^\gamma. \tag{6.11}$$

Recalling that $\psi \in ad_\gamma^*$ we see that (6.11) contradicts the definition of $\check{\lambda}_k^\gamma$. Hence, at least when $n_k = k$, equality must hold throughout (6.6), and therefore (6.5) follows as before. ∎

Hence, in agreement with Krein's result, when one of $\lambda_k(\check{\rho}_k^\gamma)$'s eigenfunctions possesses k nodal domains we find that $\check{\rho}_k^\gamma$ minimizes the first eigenvalue of each of these nodal domains. In this way, the minimization of λ_k is reduced to the earlier problem of minimizing λ_1.

Comparing this proposition to Friedland's result we note that when our result applies it conveys much more information. For a fixed γ however, Friedland's result holds for every k, while ours is only guaranteed to hold when $k = 2$. We also note that due to their reliance on the characterization of λ_k via a minimum principle, neither of our arguments may be used (directly) in characterizing $\hat{\rho}_k^\gamma$.

7. MAXIMIZING λ_k

Proposition 4.4 provides us with a pair $(\hat{\rho}_k^\gamma, \hat{u}_k^\gamma) \in ad_\gamma^* \times W_0^{1,2}(\Omega)$ for which

$$\frac{-1}{2\hat{\lambda}_k^\gamma} = \mathcal{A}_k(\hat{\rho}_k^\gamma, \hat{u}_k^\gamma) = \sup_{\rho \in ad_\gamma^*} \inf_{u \in W_0^{1,2}} \mathcal{A}_k(\rho, u). \tag{7.1}$$

In the previous section we arrived at two optimality criteria by simply reversing the order of the infima. The first criteria stated conditions for

$\check{u}_k^\gamma$ to be an eigenfunction while the second led to the valuable maximum principle then on to pointwise optimality criteria. With this in mind, we now attempt to establish the validity of

$$\mathcal{A}_k(\hat{\rho}_k^\gamma, \hat{u}_k^\gamma) = \sup_{\rho \in ad_\gamma^*} \inf_{u \in W_0^{1,2}} \mathcal{A}_k(\rho, u) = \inf_{u \in W_0^{1,2}} \sup_{\rho \in ad_\gamma^*} \mathcal{A}_k(\rho, u),$$

i.e., we wish to show that $(\hat{\rho}_k^\gamma, \hat{u}_k^\gamma)$ is a saddle point for $\mathcal{A}_k$. As with the minimizers, we shall meet with the greatest success when $k = 1$.

With regard to showing existence of saddle points for $\mathcal{A}_1$, the standard sufficient conditions, arising from the separation of convex sets in a reflexive space, require concavity of $\rho \mapsto \mathcal{A}_1(\rho, u)$ and convexity of $u \mapsto \mathcal{A}_1(\rho, u)$, neither of which holds (at least over the entire product $ad_\gamma^* \times W_0^{1,2}(\Omega)$). Weaker conditions, arising from more general fixed point theorems, involve only concavity convexity assumptions on the corresponding level sets.

Definition 7.1. *The real valued function F is said to be* **quasiconvex** *on the convex set V when the (lower) level set $\{v \in V : F(v) \leq c\}$ is convex for each real c. F is said to be* **quasiconcave** *on V when the (upper) level set $\{v \in V : F(v) \geq c\}$ is convex for each real c.*

We observe that every convex function is quasiconvex, and likewise, every concave function is quasiconcave. This notion of quasiconvexity, introduced by De Finetti around 1950, should not be confused with the notion of quasiconvexity introduced, at about the same time, by Morrey.

Lemma 7.2. *$\rho \mapsto \mathcal{A}_1(\rho, u)$ is quasiconcave on ad_γ^*.*

Proof: For $\mathcal{A}_1(\rho_i, u) \geq c,\ i = 1, 2$, we see that

$$\int_\Omega \rho_i u^2\, dx \leq (a-c)^2, \quad a = \frac{1}{2}||\nabla u||_2^2 \geq c, \quad \text{hence}$$

$$\begin{aligned}\mathcal{A}_1(t\rho_1 + (1-t)\rho_2, u) &= a - (\int_\Omega (t\rho_1 + (1-t)\rho_2)u^2\, dx)^{1/2} \\ &\geq a - (t(a-c)^2 + (1-t)(a-c)^2)^{1/2} \ = \ c. \ \blacksquare\end{aligned}$$

By $u_1(\rho)$ we shall refer to the positive first eigenfunction selected by Auchmuty's Principle. As $\mathcal{A}_1(\rho, 0) = 0$ and $u \mapsto \mathcal{A}_1(\rho, u)$ dips below zero at $u_1(\rho)$ and $-u_1(\rho)$ we see that $u \mapsto \mathcal{A}_1(\rho, u)$ is not quasiconvex on $W_0^{1,2}(\Omega)$. This difficulty is avoided however, by restricting oneself to a convex set outside a ball containing $\pm u_1(\rho)$. Such a set is the open half space exterior to $\mathcal{B}_\rho = \{u \in W_0^{1,2}(\Omega) : ||u||_\rho \leq 1/\lambda_1(\rho)\}$ and generated by the tangent space to $\partial\mathcal{B}_\rho$ at $u_1(\rho)$, i.e.,

$$\Pi_\rho = \{v \in W_0^{1,2}(\Omega) : \lambda_1^2(\rho) \int_\Omega \rho u_1(\rho) v \, dx > 1\}.$$

Lemma 7.3. *$u \mapsto \mathcal{A}_1(\rho, u)$ is strictly convex on Π_ρ.*

Proof: As $\overline{u} \in \Pi_\rho$ implies $\lambda_1(\rho) > ||\overline{u}||_\rho^{-1}$, on computing the second derivative we find that for each $v \in W_0^{1,2}(\Omega)$,

$$\begin{aligned}\langle D_2^2 \mathcal{A}_1(\rho, \overline{u}) v, v\rangle &= \int_\Omega |\nabla v|^2 \, dx - ||\overline{u}||_\rho^{-1} \int_\Omega \rho v^2 \, dx + ||\overline{u}||_\rho^{-3} \left(\int_\Omega \rho \overline{u} v \, dx\right)^2 \\ &\geq (\lambda_1(\rho) - ||\overline{u}||_\rho^{-1}) \int_\Omega \rho v^2 \, dx \geq 0 \ \blacksquare\end{aligned}$$

The saddle point theorem whose hypotheses we are aiming to fill is Theorem 3.7 of Barbu and Precupanu [3], which we restate here as

Proposition 7.4. *Let F be a real valued upper lower semicontinuous quasi concave convex function on $A \times B$. If there are $w_0 \in B$ and $c_0 < \inf_{w \in B} \sup_{v \in A} F(v, w)$ such that $\{v \in A : F(v, w_0) \geq c_0\}$ is compact, then*

$$\sup_{v \in A} \inf_{w \in B} F(v, w) = \inf_{w \in B} \sup_{v \in A} F(v, w).$$

As the set over which $u \mapsto \mathcal{A}_1(\rho, u)$ is convex depends on ρ we shall find it more convenient to consider $u \mapsto \mathcal{A}_1(\rho, u) + \pi(\rho, u)$, where $u \mapsto \pi(\rho, u)$ is the indicator function of Π_ρ, i.e.,

$$\pi(\rho, u) = \begin{cases} 0, & \text{if } u \in \Pi_\rho; \\ \infty, & \text{otherwise.} \end{cases}$$

Perturbing $\mathcal{A}_1$ by π, while rendering $u \mapsto \mathcal{A}_1(\rho, u) + \pi(\rho, u)$ convex over all of $W_0^{1,2}(\Omega)$, nonetheless retains a number of $\mathcal{A}_1$'s desired properties.

Lemma 7.5. *$\rho \mapsto \mathcal{A}_1(\rho,u)+\pi(\rho,u)$ is weak* upper semicontinuous over ad^*_γ.*

Proof: Regarding the discussion following Proposition 4.3, $\rho \mapsto \mathcal{A}_1(\rho,u)$ is weak* continuous on ad^*_γ. With respect to π, for $\overline{\rho} \in ad^*_\gamma$ we take $\{\rho^n\} \subset ad^*_\gamma$ such that $\rho^n \overset{*}{\rightharpoonup} \overline{\rho}$ and show that

$$\limsup_{n\to\infty} \pi(\rho^n,u) \le \pi(\overline{\rho},u) \qquad \forall\, u \in W_0^{1,2}(\Omega).$$

This clearly holds when $\pi(\overline{\rho},u)=\infty$, i.e., when $u \notin \Pi_{\overline{\rho}}$. If $u \in \Pi_{\overline{\rho}}$ then $\rho^n \overset{*}{\rightharpoonup} \overline{\rho}$ implies via Proposition 4.3 that $\lambda_1(\rho^n) \to \lambda_1(\overline{\rho})$ and $u_1(\rho^n) \to u_1(\overline{\rho})$ in $W_0^{1,2}(\Omega)$. As a result,

$$\lambda_1^2(\rho^n)\int_\Omega \rho^n u_1(\rho^n)u\,dx \to \lambda_1^2(\overline{\rho})\int_\Omega \overline{\rho}u_1(\overline{\rho})u\,dx.$$

$u \in \Pi_{\overline{\rho}}$ now implies that for all n greater than some constant, depending only on $\overline{\rho}$, $u \in \Pi_{\rho^n}$, i.e., $\pi(\rho^n,u)=0$. The Lemma now follows since upper semicontinuity is preserved under finite sums. ∎

On considering the sequence $\{u^n\} \subset \Pi_\rho$, $u^n = (1+\frac{1}{n})u_1(\rho)$ we find

$$\mathcal{A}_1(\rho,u^n)+\pi(\rho,u^n) = \frac{(1+1/n)^2}{2\lambda_1(\rho)} - \frac{(1+1/n)}{\lambda_1(\rho)} \to \frac{-1}{2\lambda_1(\rho)},$$

as $n\to\infty$. Consequently,

$$\begin{aligned} \inf_{u\in W_0^{1,2}} \mathcal{A}_1(\rho,u) &= \inf_{u\in W_0^{1,2}} (\mathcal{A}_1(\rho,u)+\pi(\rho,u)), \quad \text{and,} \\ \sup_{\rho\in ad^*_\gamma}\inf_{u\in W_0^{1,2}} \mathcal{A}_1(\rho,u) &= \sup_{\rho\in ad^*_\gamma}\inf_{u\in W_0^{1,2}} (\mathcal{A}_1(\rho,u)+\pi(\rho,u)). \end{aligned} \tag{7.2}$$

We also remark that, by definition,

$$\sup_{\rho\in ad^*_\gamma}\inf_{u\in W_0^{1,2}} \mathcal{A}_1(\rho,u) \le \inf_{u\in W_0^{1,2}}\sup_{\rho\in ad^*_\gamma} \mathcal{A}_1(\rho,u). \tag{7.3}$$

We can now modify the proof of Proposition 7.4 in such a way as to establish the existence of a saddle point for $\mathcal{A}_1$. This modification will require knowledge of the proof's main ingredient, the following Fixed Point Theorem of Knaster, Kuratowski, and Mazurkiewicz [17].

Proposition 7.6. *([17]) Let K be an arbitrary set in a finite dimensional separated topological linear space V. To every $v \in K$ let $\Phi(v) \subset V$ be a compact set such that the convex hull of each finite subset $\{v_1, \ldots, v_n\} \subset V$ is contained in the corresponding union $\cup_{i=1}^{n} \Phi(v_i)$. Then $\cap_{v \in K} \Phi(v) \neq \emptyset$.*

We can now prove our main result.

Proposition 7.7. $\sup\limits_{\rho \in ad_\gamma^*} \inf\limits_{u \in W_0^{1,2}} \mathcal{A}_1(\rho, u) = \inf\limits_{u \in W_0^{1,2}} \sup\limits_{\rho \in ad_\gamma^*} \mathcal{A}_1(\rho, u).$

Proof: If this is not true then (7.2) and (7.3) imply the existence of a number c for which

$$\sup_{\rho \in ad_\gamma^*} \inf_{u \in W_0^{1,2}} \mathcal{A}_1(\rho, u) + \pi(\rho, u) < c < \inf_{u \in W_0^{1,2}} \sup_{\rho \in ad_\gamma^*} \mathcal{A}_1(\rho, u). \tag{7.4}$$

This implies that if

$$\begin{gathered} A_u = \{\rho \in ad_\gamma^* : \mathcal{A}_1(\rho, u) + \pi(\rho, u) \geq c\} \quad \text{and} \\ B_\rho = \{u \in W_0^{1,2}(\Omega) : \mathcal{A}_1(\rho, u) \leq c\} \quad \text{then} \\ \bigcap_{u \in W_0^{1,2}} A_u = \bigcap_{\rho \in ad_\gamma^*} B_\rho = \emptyset \end{gathered} \tag{7.5}$$

Invoking Lemma 7.5 we find that each A_u is a weak* closed subset of the weak* compact set ad^*. Similarly, the weak lower semicontinuity of $u \mapsto \mathcal{A}_1(\rho, u)$, see [2], and the fact that c must be negative, imply that each B_ρ is a weakly closed subset of the weakly compact set $\{u \in W_0^{1,2} : ||\nabla u||_2 \leq 2\lambda_1^{-1/2}(\beta)\}$. (7.5) then implies that neither $\{A_u : u \in W_0^{1,2}(\Omega)\}$, nor $\{B_\rho : \rho \in ad_\gamma^*\}$ may possess the finite intersection property. Hence, there must exist $\{u_1, \ldots, u_n\} \subset W_0^{1,2}(\Omega)$ and $\{\rho_1, \ldots, \rho_m\} \subset ad_\gamma^*$ such that

$$\bigcap_{i=1}^{n} A_{u_i} = \bigcap_{i=1}^{m} B_{\rho_i} = \emptyset. \tag{7.6}$$

We now invoke Proposition 7.6 with the goal of contradicting (7.6). Consider $A' = \mathrm{co}\{\rho_i\}_{i=1}^m$ and $B' = \mathrm{co}\{u_i\}_{i=1}^n$, where co denotes convex hull, and the map $\Phi : A' \times B' \to A' \times B'$ of the form,

$$\Phi(\phi, v) = \{(\rho, u) \in A' \times B' : \mathcal{A}_1(\rho, v) + \pi(\rho, v) \geq c \text{ or } \mathcal{A}_1(\phi, u) \leq c\}.$$

Note that A' and B', being finitely generated, are compact. We claim that Φ satisfies the hypotheses of Proposition 7.6. That the image of Φ is not empty follows from $(\phi, v) \in \Phi(\phi, v)$. That $\Phi(\phi, v)$ is compact follows from $\Phi(\phi, v) = (A_v \cap A') \times B' \cup A' \times (B_\phi \cap B')$. Moreover, $\{(\phi_i, v_i)\}_{i=1}^p \subset A' \times B'$ and $t_i \geq 0$, $\sum_{i=1}^p t_i = 1$ such that

$$\sum_{i=1}^p t_i(\phi_i, v_i) \notin \Phi(\phi_j, v_j)\ j = 1, \ldots, p, \quad \text{implies that}$$

$$\mathcal{A}_1(\sum_{i=1}^p t_i\phi_i, v_j) + \pi(\sum_{i=1}^p t_i\phi_i, v_j) < c \quad j = 1, \ldots, p, \text{ and} \tag{7.7}$$

$$\mathcal{A}_1(\phi_j, \sum_{i=1}^p t_i v_i) > c \quad j = 1, \ldots, p. \tag{7.8}$$

Setting $\overline{v} = \sum_{i=1}^p t_i v_i$ and $\overline{\phi} = \sum_{i=1}^p t_i\phi_i$ we see that (7.7), in view of the convexity of $u \mapsto \mathcal{A}_1(\overline{\phi}, u) + \pi(\overline{\phi}, u)$, yields $\mathcal{A}_1(\overline{\phi}, \overline{v}) < c$. As this is contradicted by (7.8) in conjunction with the quasiconcavity of $\rho \mapsto \mathcal{A}_1(\rho, \overline{v})$ we may conclude that Φ satisfies the hypotheses of Proposition 7.6. As a result, there exists a pair $(\overline{\rho}, \overline{u}) \in A' \times B'$ such that $(\overline{\rho}, \overline{u}) \in \Phi(\rho, u)$ for all $(\rho, u) \in A' \times B'$, i.e.,

$$\mathcal{A}_1(\overline{\rho}, u) + \pi(\overline{\rho}, u) \geq c \ \text{ or } \ \mathcal{A}_1(\rho, \overline{u}) \leq c \qquad \forall\ (\rho, u) \in A' \times B'.$$

This implies however, that either

$$\overline{\rho} \in \bigcap_{i=1}^n A_{u_i} \quad \text{or} \quad \overline{u} \in \bigcap_{i=1}^m B_{\rho_i},$$

a clear contradiction of (7.6). ∎

This result, together with (7.1), implies that $(\hat{\rho}^\gamma, \hat{u}^\gamma)$ is a saddle point for $\mathcal{A}_1$. The resulting optimality criteria now take the form

$$\mathcal{A}_1(\rho, \hat{u}^\gamma) \le \mathcal{A}_1(\hat{\rho}^\gamma, \hat{u}^\gamma) \le \mathcal{A}_1(\hat{\rho}^\gamma, u) \qquad \forall\, (\rho, u) \in ad^*_\gamma \times W_0^{1,2}(\Omega).$$

The second inequality states that $\hat{u}^\gamma$ is an eigenfunction corresponding to $\hat{\rho}^\gamma$, while the first is the minimum principle

$$\int_\Omega \hat{\rho}^\gamma (\hat{u}^\gamma)^2\, dx \le \int_\Omega \rho (\hat{u}^\gamma)^2\, dx \qquad \forall\, \rho \in ad^*_\gamma. \tag{7.9}$$

Again, without loss, we assume that $\hat{u}^\gamma(x) > 0$ in Ω. An argument identical to that of Proposition 6.1 deduces from (7.9) analogous optimality criteria.

Proposition 7.8. *There exists an $\hat{\ell} \ge 0$ such that for each $x \in \Omega$,*

$$\begin{aligned} \hat{\rho}^\gamma(x) = \beta &\Rightarrow \hat{u}^\gamma(x) \le \hat{\ell} \\ \alpha < \hat{\rho}^\gamma(x) < \beta &\Rightarrow \hat{u}^\gamma(x) = \hat{\ell} \\ \hat{\rho}^\gamma(x) = \alpha &\Rightarrow \hat{u}^\gamma(x) \ge \hat{\ell} \end{aligned} \tag{7.10}$$

As in the previous section, the optimal design takes values between α and β on a level set of the corresponding eigenfunction. This leads to

Proposition 7.9. *If $\partial\Omega$ is C^2 then $\hat{\rho}^\gamma \in ad_\gamma$ and is unique.*

Proof: The C^2 hypothesis on $\partial\Omega$ implies via (7.10) that every maximizer of $\rho \mapsto \lambda_1(\rho)$ on ad^*_γ is an element of ad_γ. Recalling however, the convexity of $\rho \mapsto 1/\lambda_1(\rho)$, Lemma 3.1, we find that the maximizers of $\rho \mapsto \lambda_1(\rho)$ form a convex subset of ad^*_γ. Our claim now follows, as the only convex set of extreme points is a singleton. ∎

Exploiting the structure of the set of saddle points for $\mathcal{A}_1$ we can remove the smooth boundary hypothesis.

Proposition 7.10. *$\hat{\rho}^\gamma \in ad_\gamma$ and is unique.*

Proof: From (7.9) we see that $\hat{\rho}^\gamma$ minimizes the weak* continuous positive linear functional $\rho \mapsto \int_\Omega \rho(\hat{u}^\gamma)^2\,dx$ over the convex weak* compact set ad_γ^*. Hence, $\hat{\rho}^\gamma$ maximizes the weak* continuous convex functional $\rho \mapsto (\int_\Omega \rho(\hat{u}^\gamma)^2\,dx)^{-1}$ over ad_γ^*. Recalling [5] II.54, we may take $\hat{\rho}^\gamma \in ad_\gamma$. Now assume the existence of another maximizer $\psi \in ad_\gamma^*$, i.e., $\hat{\lambda}_1^\gamma = \lambda_1(\psi)$. We recall from Ekeland and Temam [9] VI.1.4, that the set of saddle points of a bivariate function on a product space is itself a product. As a result, $(\psi, \hat{u}^\gamma)$ is also a saddle point for $\mathcal{A}_1$. The fact then, that $\hat{\lambda}_1^\gamma$ and $\hat{u}^\gamma$ are the first eigenvalue and eigenfunction for both $\hat{\rho}^\gamma$ and ψ implies, via the weak formulation, that $\int_\Omega \hat{\rho}^\gamma \hat{u}^\gamma v\,dx = \int_\Omega \psi \hat{u}^\gamma v\,dx$ for every $v \in W_0^{1,2}(\Omega)$, from which we conclude that $\hat{\rho}^\gamma = \psi$ a.e. in Ω. ∎

The results of this section have been highly dependent on Auchmuty's Principle. From the minimum principle, (7.9), however it follows immediately that

$$\mathcal{R}(\rho, \hat{u}^\gamma) \leq \mathcal{R}(\hat{\rho}^\gamma, \hat{u}^\gamma) \leq \mathcal{R}(\hat{\rho}^\gamma, u) \qquad \forall\,(\rho, u) \in ad_\gamma^* \times \left(W_0^{1,2}(\Omega) \setminus \{0\}\right).$$

That is, $(\hat{\rho}^\gamma, \hat{u}^\gamma)$ is a saddle point for the Rayleigh quotient $\mathcal{R}$. To reveal the deeper connections between Rayleigh's and Auchmuty's Principle we now proceed with a more careful derivation of the previous observation.

We shall find it convenient to restrict ourselves to the positive elements of $W_0^{1,2}(\Omega)$, $\mathcal{P} = \{u \in W_0^{1,2}(\Omega) : u(x) > 0\}$, and in some contexts even to positive rays, $\mathbf{R}^+u = \{tu : t > 0\}$ for $u \in \mathcal{P}$. We will also need the following equivalent formulation of quasiconvexity.

Proposition 7.11. *(Roberts and Varberg [22] 81B) If $v \mapsto F(v)$ is C^1 over the convex set V then it is quasiconvex on V if and only if*

$$F(u) \leq F(v) \Rightarrow \langle DF(v), u - v\rangle \leq 0, \qquad \forall\, u, v \in V.$$

Lemma 7.12. *$u \mapsto \mathcal{A}_1(\rho, u)$ is quasiconvex along positive rays.*

Proof: $u, v \in W_0^{1,2}(\Omega)$ such that $\mathcal{A}_1(\rho, u) \leq \mathcal{A}_1(\rho, v)$ implies

$$\|v\|_\rho \leq \tfrac{1}{2}\|\nabla v\|_2^2 - \tfrac{1}{2}\|\nabla u\|_2^2 + \|u\|_\rho, \quad \text{hence}$$

$$\begin{aligned}
\langle D_2\mathcal{A}_1(\rho, v), u - v\rangle &= \int_\Omega \nabla v \cdot \nabla u \, dx - \int_\Omega |\nabla v|^2 \, dx \\
&\qquad - \|v\|_\rho^{-1} \int_\Omega \rho v u \, dx + \|v\|_\rho \\
&\leq -\tfrac{1}{2}\|\nabla v\|_2^2 + \int_\Omega \nabla v \cdot \nabla u \, dx - \tfrac{1}{2}\|\nabla u\|_2^2 \\
&\qquad + \|u\|_\rho - \|v\|_\rho^{-1} \int_\Omega \rho v u \, dx \\
&= -\tfrac{1}{2}\|\nabla u - \nabla v\|_2^2
\end{aligned}$$

when $u, v \in \mathbf{R}^+ u_0$ for some $u_0 \in \mathcal{P}$. ∎

We are now able to invoke Proposition 7.4 without modification and conclude that for each $u \in \mathcal{P}$

$$\sup_{\rho \in ad_\gamma^*} \inf_{t \geq 0} \mathcal{A}_1(\rho, tu) = \inf_{t \geq 0} \sup_{\rho \in ad_\gamma^*} \mathcal{A}_1(\rho, tu). \tag{7.11}$$

Denoting differentiation with respect to t by $'$ we find $\mathcal{A}_1'(\rho, tu) = t\|\nabla u\|_2^2 - \|u\|_\rho$ and $\mathcal{A}_1''(\rho, tu) = \|\nabla u\|_2^2$. Hence, $t \mapsto \mathcal{A}_1(\rho, tu)$ is minimized at

$$t_0 = \frac{\|u\|_\rho}{\|\nabla u\|_2^2}, \quad \text{for which } \mathcal{A}_1(\rho, t_0 u) = -\frac{1}{2}\frac{\|u\|_\rho^2}{\|\nabla u\|_2^2} = -\tfrac{1}{2}\mathcal{R}^{-1}(\rho, u).$$

With this observation, (7.11) becomes

$$\sup_{\rho \in ad_\gamma^*} -\tfrac{1}{2}\mathcal{R}^{-1}(\rho, u) = \inf_{t \geq 0} \sup_{\rho \in ad_\gamma^*} \mathcal{A}_1(\rho, tu).$$

Taking the infimum of each side over $\mathcal{P}$ we arrive at

$$\inf_{u \in \mathcal{P}} \sup_{\rho \in ad_\gamma^*} -\tfrac{1}{2}\mathcal{R}^{-1}(\rho, u) = \inf_{u \in \mathcal{P}} \sup_{\rho \in ad_\gamma^*} \mathcal{A}_1(\rho, u).$$

On applying Proposition 7.7 then Auchmuty's and finally Rayleigh's Principle we arrive at the following string of equalities.

$$\begin{gathered}\sup_{u\in\mathcal{P}}\inf_{\rho\in ad_\gamma^*}\mathcal{R}^{-1}(\rho,u)=-2\inf_{u\in\mathcal{P}}\sup_{\rho\in ad_\gamma^*}\mathcal{A}_1(\rho,u)=\\ -2\sup_{\rho\in ad_\gamma^*}\inf_{u\in\mathcal{P}}\mathcal{A}_1(\rho,u)=-2\sup_{\rho\in ad_\gamma^*}\frac{-1}{2\lambda_1(\rho)}=\\ \inf_{\rho\in ad_\gamma^*}\frac{1}{\lambda_1(\rho)}=\inf_{\rho\in ad_\gamma^*}\sup_{u\in\mathcal{P}}\mathcal{R}^{-1}(\rho,u).\end{gathered}\tag{7.12}$$

As the common value in (7.12) is $1/\hat{\lambda}_1^\gamma$ and $\mathcal{R}^{-1}(\hat{\rho}^\gamma,\hat{u}^\gamma)=1/\hat{\lambda}_1^\gamma$ we find

Proposition 7.13. *$(\hat{\rho}^\gamma,\hat{u}^\gamma)$ is a saddle point for $\mathcal{R}^{-1}$ on $ad_\gamma^*\times W_0^{1,2}(\Omega)$.*

In order to characterize $\hat{\rho}_k^\gamma$ for $k>1$ through similar methods, more work is clearly needed on the geometric properties of $\rho\mapsto P_{k-1}(\rho)$, the map from densities to projections.

8. SYMMETRY CONSIDERATIONS

The optimality conditions established in §7 prescribe $\hat{\rho}^\gamma$ as a function of $\hat{u}^\gamma$. Although $\hat{u}^\gamma$ is unknown, inspired by the ambitious work of Kawohl [15], we demonstrate how various geometric properties of Ω are preserved by $\hat{u}^\gamma$ and therefore $\hat{\rho}^\gamma$.

We shall assume throughout this section that Ω is a bounded domain in $\mathbf{R}^N$ with C^2 boundary. We capture the geometric properties of such sets in **optimal caps**. Beginning with a hyperplane $T_\mu=\{x\in\mathbf{R}^N:\mathbf{n}\cdot x=\mu\}$, where $\mathbf{n}$ is a unit vector in $\mathbf{R}^N$, we note that for μ larger than some $\overline{\mu}$, $T_\mu\cap\overline{\Omega}=\emptyset$. Let μ_0 denote the smallest μ for which $T_\mu\cap\overline{\Omega}\neq\emptyset$. As we diminish μ below μ_0, T_μ will cut off from Ω an open cap $\Sigma(\mu)$, the part of Ω on the same side of T_μ as $T_{\overline{\mu}}$. Let $\Sigma'(\mu)$ denote the reflection of $\Sigma(\mu)$ across T_μ. In the beginning $\Sigma'(\mu)$ will be contained in Ω, and there will be a smallest μ such that the reflected cap $\Sigma'(\mu)$ is contained in Ω. Setting $\mu_2=\inf\{\mu:\Sigma'(\mu)\subseteq\Omega\}$ we arrive at

Definition 8.1. *$\Sigma(\mu_2)$ is the optimal cap in direction $\mathbf{n}$ of Ω.*

For ease of exposition we will assume in what follows that $\mathbf{n}$ designates the x_1 direction, i.e., $\mathbf{n} = (1, 0, \ldots, 0)$. For $x \in \Omega$ we shall denote by x^μ its reflection in the plane T_μ. With these conventions, Gidas, Ni, and Nirenberg [11], relying only on the Hopf Boundary Point Lemma, proved the following beautiful result

Proposition 8.2. *([11] Theorem 3.1) If $\overline{u}$ is a positive $C^2(\overline{\Omega})$ solution of*

$$-\Delta u = f(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

where $f : \mathbf{R} \to \mathbf{R}$ is locally Lipschitz, then for $\mu_2 < \mu < \mu_0$

$$\frac{\partial \overline{u}}{\partial x_1}(x) < 0 \quad \text{and} \quad \overline{u}(x) < \overline{u}(x^\mu) \qquad \forall\, x \in \Sigma(\mu).$$

The consequences of such a result are many. For example,

Corollary 8.3. *([11] Corollary 1) If Ω is convex and symmetric in the x_1 direction then, for each $c \in \mathbf{R}$, so is $\{x \in \Omega : \overline{u}(x) \geq c\}$.*

This provides us with ample motivation. For the set on which $\hat{\rho}^\gamma = \alpha$ is precisely of the form $\{x \in \Omega : \hat{u}^\gamma(x) \geq \hat{\ell}\}$! In view of (7.10) we see that $\hat{u}^\gamma$ is a positive $C^1(\overline{\Omega})$ solution of

$$-\Delta u = \hat{\lambda}_1^\gamma f(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \text{where}$$
$$f(t) = \beta t H(t) + (\alpha - \beta) t H(t - \hat{\ell}), \quad H \text{ being the standard}$$
$$\text{Heaviside function,} \quad H(t) = \begin{cases} 1, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0. \end{cases}$$

Looking back at Proposition 8.2, the first thing we notice is that our f is discontinuous. We prescribe a simple remedy.

(i) Approximate f by a sequence $\{f_\eta\}$ of Lipschitz regularizations.

(ii) Solve $-\Delta u = f_\eta(u)$ in Ω, $u|_{\partial\Omega} = 0$, for a $C^2(\overline{\Omega})$ positive u_η.

(iii) Find a subsequence $\{u_{\eta_j}\}$ which converges in $C^1(\overline{\Omega})$ to $\hat{u}^\gamma$.

One can then show that $\hat{u}^\gamma$ satisfies weak versions of Proposition 8.2 and its corollary. Our approach to (ii) is variational, i.e., we demonstrate that u_η minimizes a relevant nonlinear functional. (iii) will then follow from the epi convergence of these functionals. We begin with the limiting functional,

$$\mathcal{J}(u) = \int_\Omega \tfrac{1}{2}|\nabla u|^2 - \hat{\lambda}_1^\gamma F(u)\,dx, \qquad F(s) = \int_0^s f(t)\,dt.$$

Lemma 8.4. *$\hat{u}^\gamma$ is the unique nontrivial minimizer of $u \mapsto \mathcal{J}(u)$ on $W_0^{1,2}(\Omega)$.*

Proof: We first establish existence. The fact that F is continuous and satisfies

$$|F(t)| \le \frac{\hat{\ell}^2}{2}(\beta - \alpha) + \frac{\alpha}{2}t^2,$$

implies, through a well known result of Krasnosel'skii, see Ekeland and Temam [9] IV.1.1, that $u(x) \mapsto F(u(x))$ is continuous from $L^2(\Omega)$ to $L^1(\Omega)$. This in turn implies, as a consequence of the compact imbedding $W_0^{1,2}(\Omega) \to L^2(\Omega)$, the weak $W_0^{1,2}(\Omega)$ continuity of $u \mapsto \int_\Omega F(u)\,dx$. This observation in conjunction with the weak $W_0^{1,2}(\Omega)$ lower semicontinuity of $u \mapsto \int_\Omega |\nabla u|^2\,dx$ yields the desired weak $W_0^{1,2}(\Omega)$ lower semicontinuity of $u \mapsto \mathcal{J}(u)$. Note that

$$\begin{aligned} 2\mathcal{J}(u) &= \int_\Omega |\nabla u|^2\,dx - \hat{\lambda}_1^\gamma \alpha \int_{u \ge \hat{\ell}} u^2\,dx - \hat{\lambda}_1^\gamma \beta \int_{0<u<\hat{\ell}} u^2\,dx \\ &\ge \left(1 - \frac{\hat{\lambda}_1^\gamma \alpha}{\Lambda}\right) \int_\Omega |\nabla u|^2\,dx - \hat{\lambda}_1^\gamma \beta |\Omega| \hat{\ell}^2, \end{aligned}$$

where Λ is the first eigenvalue of the corresponding homogeneous membrane

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

In order to conclude that $\mathcal{J}$ is coercive we need only demonstrate that $\hat{\lambda}_1^\gamma \alpha < \Lambda$. But this is an immediate consequence of $\hat{\lambda}_1^\gamma \le \lambda_1(\alpha)$ and

$\lambda_1(\alpha) = \Lambda/\alpha$. Now $\mathcal{J}$, being a coercive weakly lower semicontinuous function on $W_0^{1,2}(\Omega)$, attains its minimum at some $\overline{u}$. Note that if $\overline{u}$ were not nonnegative we would face the contradiction $\mathcal{J}(|\overline{u}|) < \mathcal{J}(\overline{u})$.

Recall that a function dominated by the linear function with slope Λ in a neighborhood of ∞ is said to be sublinear with respect to Ω. That f is sublinear follows from $\limsup_{t\to\infty} f(t)t^{-1} = \alpha$. This is precisely the sufficient condition used in Stuart and Toland [23] 3.2, from which we may deduce that $\overline{u}$ is a $W^{2,2}(\Omega)$ solution of the associated Euler inclusion

$$-\Delta u \in \hat{\lambda}_1^\gamma f_\star(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \text{where} \tag{8.1}$$
$$f_\star(t) = \begin{cases} \{f(t)\}, & \text{if } t \neq \hat{\ell}; \\ [\alpha\hat{\ell}, \beta\hat{\ell}], & \text{if } t = \hat{\ell}. \end{cases}$$

Arguing as in Lemma 3.6(i) we find that the measure of $\{x \in \Omega : \overline{u}(x) = \hat{\ell}\}$ is zero. As a result, (8.1) is equivalent to

$$-\Delta \overline{u} = \hat{\lambda}_1^\gamma \overline{\rho}\,\overline{u} \quad \text{in } \Omega, \quad \overline{u}|_{\partial\Omega} = 0, \quad \text{where} \tag{8.2}$$
$$\overline{\rho}(x) = \begin{cases} \alpha, & \text{if } \overline{u}(x) \geq \hat{\ell}; \\ \beta, & \text{if } \overline{u}(x) \leq \hat{\ell}. \end{cases}$$

Recalling the optimality conditions of (7.10) we see that $\overline{\rho}$ maximizes $\rho \mapsto \lambda_1(\rho)$ on $ad^*_{\overline{\gamma}}$, $\overline{\gamma} = |\{x \in \Omega : \overline{u}(x) \geq \hat{\ell}\}|$. Hence, $-\Delta\overline{u} = \hat{\lambda}_1^{\overline{\gamma}}\overline{\rho}\,\overline{u}$ in Ω. Comparing this with (8.2) while recalling the monotonicity of $\gamma \mapsto \hat{\lambda}_1^\gamma$, Proposition 5.8(ii) we find $\overline{\gamma} = \gamma$, so that $\overline{\rho}$ is indeed the unique $\hat{\rho}^\gamma$. Between this and (8.2), $\overline{u}$ can be nothing else than $\hat{u}^\gamma$. ∎

In the interest of obtaining a uniform $C^1(\overline{\Omega})$ bound on $\{u_\eta\}$ we truncate f prior to its regularization.

$$\tilde{f}(t) = \begin{cases} f(t), & \text{if } t < 1 + \|\hat{u}^\gamma\|_{L^\infty}; \\ -1, & \text{if } t \geq 1 + \|\hat{u}^\gamma\|_{L^\infty}. \end{cases}$$

Remark 8.5. *With respect to the associated*

$$\tilde{\mathcal{J}}(u) = \int_\Omega \tfrac{1}{2}|\nabla u|^2 - \hat{\lambda}_1^\gamma \tilde{F}(u)\,dx, \quad \tilde{F}(s) = \int_0^s \tilde{f}(t)\,dt$$

we note that $\mathcal{J}(u) \le \tilde{\mathcal{J}}(u)$ *for every* $u \in W_0^{1,2}(\Omega)$, *yet* $\tilde{\mathcal{J}}(\hat{u}^\gamma) = 0$. *Hence,* $\hat{u}^\gamma$ *is the unique nontrivial minimizer of* $\tilde{\mathcal{J}}$.

We employ the following regularization of $\tilde{f}$, see Attouch [1] 2.64,

Lemma 8.6. *The Moreau-Yosida approximation of index* η *of* $\tilde{f}$ *is the function*

$$\tilde{f}_\eta(t) = \inf_{s\in\mathbf{R}} \{\tilde{f}(s) + \frac{1}{2\eta}|t-s|^2\}.$$

For each $\eta > 0$ *the function* $t \mapsto \tilde{f}_\eta(t)$ *is locally Lipschitz, while for each* $t \in \mathbf{R}$ *the sequence* $\{\tilde{f}_\eta(t)\}$ *increases monotonically to* $\tilde{f}(t)$ *as* η *decreases to* 0, *i.e.,*

$$\tilde{f}(t) = \lim_{\eta\downarrow 0} \tilde{f}_\eta(t) = \sup_{\eta>0} \tilde{f}_\eta(t). \tag{8.3}$$

Remark 8.7. *As* $\tilde{f}$ *is already Lipschitz on* $(-\infty, \hat{\ell})$ *there exists an* $\eta' > 0$ *and* $t' > \hat{\ell}/2$ *such that* $\tilde{f}_\eta(t) = \tilde{f}(t)$ *when* $\eta < \eta'$ *and* $t < t'$.

We consider the associated nonlinear functional

$$\tilde{\mathcal{J}}_\eta(u) = \int_\Omega \tfrac{1}{2}|\nabla u|^2 - \lambda_\eta \tilde{F}_\eta(u)\,dx, \quad \text{where}$$
$$\tilde{F}_\eta(s) = \int_0^s \tilde{f}_\eta(t)\,dt, \quad \text{and} \quad \lambda_\eta = \frac{1+\eta}{2\hat{\lambda}_1^\gamma \int_\Omega \tilde{F}_\eta(\hat{u}^\gamma)\,dx}.$$

Part (ii) of our program is carried out in

Lemma 8.8. *There exists a positive* $C^2(\overline{\Omega})$ *function* u_η *which minimizes* $\tilde{\mathcal{J}}_\eta$ *over* $W_0^{1,2}(\Omega)$. *This* u_η *is a classical solution of the nonlinear boundary value problem*

$$-\Delta u = \lambda_\eta \tilde{f}_\eta(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{8.4}$$

Proof: The existence of a $W_0^{1,2}(\Omega)$ minimizer follows from the exact same procedure used in Lemma 8.4. We begin by demonstrating that this minimizer can not be identically zero. As $\tilde{\mathcal{J}}_\eta(0) = 0$ we have only to exhibit a

$W_0^{1,2}(\Omega)$ function for which $\tilde{\mathcal{J}}_\eta$ becomes negative. Choosing $\hat{u}^\gamma$,

$$\tilde{\mathcal{J}}_\eta(\hat{u}^\gamma) = \int_\Omega \tfrac{1}{2}|\nabla \hat{u}^\gamma|^2 - \lambda_\eta \tilde{F}_\eta(\hat{u}^\gamma)\,dx = \frac{1}{2\hat{\lambda}_1^\gamma} - \frac{1+\eta}{2\hat{\lambda}_1^\gamma} = \frac{-\eta}{2\hat{\lambda}_1^\gamma} < 0.$$

This should help explain our choice of λ_η. Now if u_η is a $W_0^{1,2}(\Omega)$ minimizer of $\tilde{\mathcal{J}}_\eta$ then u_η is a critical point of $\tilde{\mathcal{J}}_\eta$, i.e., a weak solution of the associated Euler equation, (8.4). That u_η is actually a classical solution follows from the bounded Lipschitz nature of $\tilde{f}_\eta$. For details, follow [12] 9.15 with 7.26 and 9.19.

Regarding the sign of u_η, consider the open set $A_\eta = \{x \in \Omega : u_\eta(x) < 0\}$. Noticing that $\tilde{f}_\eta(t) = 0$ for $t \leq 0$ we see that (8.4) implies that $\Delta u_\eta = 0$ in A_η and $u_\eta = 0$ on ∂A_η. The maximum principle now implies that $u_\eta \equiv 0$ in A_η so $A_\eta = \emptyset$, i.e., $u_\eta(x) \geq 0$ in Ω. Applying the maximum principle once more reveals that u_η can not have an interior zero without being identically zero. As u_η has been shown to be nontrivial we conclude $u_\eta(x) > 0$ in Ω. ∎

We now aim at isolating a nontrivial cluster point of $\{u_\eta\}$.

Lemma 8.9. *There exists a subsequence $\{u_{\eta_j}\} \subseteq \{u_\eta\}$ and a $C^1(\overline{\Omega})$ function $\overline{u} \not\equiv 0$ such that $u_{\eta_j} \to \overline{u}$ in $C^1(\overline{\Omega})$ as $\eta_j \to 0$.*

Proof: The standard $L^p(\Omega)$ estimate, [12] 9.17, gives a $c(p)$, independent of η, for which $\eta < \eta'$ implies

$$\begin{aligned} \|u_\eta\|_{W^{2,p}} &\leq c(p)\|\lambda_\eta \tilde{f}_\eta\|_{L^p} \leq c(p)\lambda_\eta \|\tilde{f}_\eta\|_{L^\infty} \leq c(p)\lambda_{\eta'}\|\tilde{f}\|_{L^\infty} \\ &\leq c(p)\lambda_{\eta'}\beta(1 + \|\hat{u}^\gamma\|_{L^\infty}). \end{aligned}$$

As $W^{2,p}(\Omega)$ is reflexive, this uniform bound implies the existence of a sequence $\{u_{\eta_j}\} \subseteq \{u_\eta\}$ and a function $\overline{u} \in W^{2,p}(\Omega)$ such that $u_{\eta_j} \rightharpoonup \overline{u}$ in $W^{2,p}(\Omega)$ as $\eta_j \to 0$. Recalling that the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ is compact for $p > N$, we conclude that $u_{\eta_j} \to \overline{u}$ in $C^1(\overline{\Omega})$. $u_{\eta_j} > 0$ in Ω

clearly implies that $\overline{u} \geq 0$. Now if $\overline{u}$ is identically zero then for all η_j below some $\bar{\eta}$ we must have $u_{\eta_j} < t'$ a.e. in Ω. As a result $\tilde{f}_{\eta_j}(u_{\eta_j}(x)) = \tilde{f}(u_{\eta_j}(x)) = \beta u_{\eta_j}(x)$ a.e. in Ω. With this however, (8.3) is simply

$$-\Delta u_{\eta_j} = \lambda_{\eta_j} \beta u_{\eta_j} \quad \text{in } \Omega, \quad u_{\eta_j}|_{\partial\Omega} = 0.$$

As $u_{\eta_j} > 0$ in Ω this implies that $\lambda_{\eta_j} = \lambda_1(\beta)$, contradicting

$$\lambda_{\eta_j} = \frac{1+\eta_j}{2\hat{\lambda}_1^\gamma \int_\Omega \tilde{F}_{\eta_j}(\hat{u}^\gamma)\,dx} \geq \frac{1+\eta_j}{2\hat{\lambda}_1^\gamma \int_\Omega \tilde{F}(\hat{u}^\gamma)\,dx} = (1+\eta_j)\hat{\lambda}_1^\gamma > \hat{\lambda}_1^\gamma \geq \lambda_1(\beta).$$

Consequently, $\overline{u} \not\equiv 0$. ∎

That this $\overline{u}$ actually minimizes $u \mapsto \tilde{\mathcal{J}}(u)$, will follow from the epi convergence of $\{\tilde{\mathcal{J}}_{\eta_j}\}$. The computation of the epi limit is facilitated by

Proposition 8.10. *(Attouch [1] 2.46) Let $\{\mathcal{F}^n\}$ be a monotone decreasing sequence of functions from V, a general space, into $\overline{\mathbf{R}}$, with pointwise limit $\mathcal{F}$. Then, for every topology τ on V, τ–$\lim_{\mathrm{e}} \mathcal{F}^n = \mathrm{cl}_\tau \mathcal{F}$.*

Here, $\mathrm{cl}_\tau \mathcal{F}$ denotes the largest τ lower semicontinuous function nowhere greater than $\mathcal{F}$. Denoting the strong topology by s, we have

Lemma 8.11. *s–$\lim_{\mathrm{e}} \tilde{\mathcal{J}}_{\eta_j} = \tilde{\mathcal{J}}$ in $W_0^{1,2}(\Omega)$.*

Proof: We write $\tilde{\mathcal{J}}_{\eta_j}(u) = \mathcal{J}^1(u) + \lambda_{\eta_j} \tilde{\mathcal{J}}^2_{\eta_j}(u)$, where

$$\mathcal{J}^1(u) = \tfrac{1}{2}\int_\Omega |\nabla u|^2\,dx \quad \text{and} \quad \tilde{\mathcal{J}}^2_{\eta_j}(u) = -\int_\Omega \tilde{F}_{\eta_j}(u)\,dx.$$

We now exploit the monotonicity of the Moreau-Yosida approximation scheme. For (8.3) implies, after subsequent applications of the Monotone Convergence Theorem, that

$$\inf_{\eta_j > 0} \tilde{\mathcal{J}}^2_{\eta_j}(u) = \tilde{\mathcal{J}}^2(u) \equiv -\int_\Omega \tilde{F}(u)\,dx.$$

As $u \mapsto \tilde{\mathcal{J}}^2(u)$ is continuous in the strong topology on $W_0^{1,2}(\Omega)$ we deduce from the previous proposition that $s\text{--}\lim_{\mathrm{e}} \tilde{\mathcal{J}}^2_{\eta_j} = \tilde{\mathcal{J}}^2$ in $W_0^{1,2}(\Omega)$. In addition the fact that $\{\lambda_{\eta_j}\}$ decreases to $\hat{\lambda}_1^\gamma$, implies that $s\text{--}\lim_{\mathrm{e}} \lambda_{\eta_j} \tilde{\mathcal{J}}^2_{\eta_j} = \hat{\lambda}_1^\gamma \tilde{\mathcal{J}}^2$ in $W_0^{1,2}(\Omega)$. Noting the strong $W_0^{1,2}(\Omega)$ continuity of $u \mapsto \mathcal{J}^1(u)$ we need only recall Proposition 5.3 to conclude that

$$s\text{--}\lim_{\mathrm{e}} (\mathcal{J}^1 + \lambda_{\eta_j} \tilde{\mathcal{J}}^2_{\eta_j}) = \mathcal{J}^1 + \hat{\lambda}_1^\gamma \tilde{\mathcal{J}}^2 \quad \text{in } W_0^{1,2}(\Omega). \blacksquare$$

The main result of this section now follows.

Proposition 8.12. *For* $\mu_2 < \mu < \mu_0$

$$\frac{\partial \hat{u}^\gamma}{\partial x_1}(x) \leq 0 \quad \textit{and} \quad \hat{u}^\gamma(x) \leq \hat{u}^\gamma(x^\mu) \qquad \forall\, x \in \Sigma(\mu).$$

Proof: $u_{\eta_j} \to \overline{u}$ in $C^1(\overline{\Omega})$ implies $u_{\eta_j} \to \overline{u}$ in $W_0^{1,2}(\Omega)$. The strong epi convergence of the $\tilde{\mathcal{J}}_{\eta_j}$ to $\tilde{\mathcal{J}}$ now implies, via Proposition 5.2, that $\overline{u}$ is a nontrivial minimizer of $\tilde{\mathcal{J}}$. Remark 8.4 however, informs us that $\tilde{\mathcal{J}}$ has but one nontrivial minimizer, namely $\hat{u}^\gamma$. Hence, $u_{\eta_j} \to \hat{u}^\gamma$ in $C^1(\overline{\Omega})$. As u_{η_j} satisfies Proposition 8.2 for each η_j, our result is an immediate consequence of their $C^1(\overline{\Omega})$ convergence to $\hat{u}^\gamma$. $\blacksquare$

We remark that Corollary 8.3 applies without modification. Hence, in the spirit of Kawohl's Corollary 3.2 [15], we arrive at the following consequence of Proposition 8.12.

Corollary 8.13. *If Ω is convex and symmetric in N orthogonal directions then the set $\{x \in \Omega : \hat{\rho}^\gamma(x) = \alpha\}$ is convex and symmetric in these directions, and is star-shaped with respect to the center of symmetry.*

Proof: Apply Corollary 8.3 in each of these N directions and recall the optimality conditions of (7.10). $\blacksquare$

We remark that Cosner and Schmitt [8] have recently shown that the level sets are indeed star-shaped when 'orthogonal' is replaced by 'linearly independent' in the above hypothesis.

Recalling that the optimality conditions for $\check{\rho}^\gamma$ are near mirror images of those for $\hat{\rho}^\gamma$, we expect analogous results to hold for the set $\{x \in \Omega : \check{\rho}^\gamma(x) = \beta\}$.

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