EXTREMAL EIGENVALUE PROBLEMS FOR TWO PHASE CONDUCTORS

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1. Introduction

Our conductor is to be a mixture of the two materials with respective conductivities, specific heats, and densities, c_i, κ_i, and ρ_i, $i = 1, 2$. We consolidate the latter two parameters in $\varrho_i = \kappa_i \rho_i$. The body is to fill the region $\Omega \subset \mathbf{R}^d$ subject to the constraint that the volume of conductor one is γ. We suppose that conductor one occupies the set $\omega \subset \Omega$ and denote by χ_ω the characteristic function of ω, i.e., $\chi_\omega(x) = 1$ if $x \in \omega$ and $\chi_\omega(x) = 0$ otherwise. The temperature, $u(x,t)$, of the conductor, assumed encased in an ice–bath, is the solution of the initial boundary value problem

$$
\begin{aligned}
\varrho_+(\chi_\omega) u_t(x,t) - \nabla \cdot c_+(\chi_\omega) \nabla u(x,t) &= 0, \quad x \in \Omega,\ t > 0, \\
u(x,t) &= 0, \quad x \in \partial\Omega,\ t > 0, \\
u(x,0) &= u_0(x), \quad x \in \Omega,
\end{aligned}
\tag{1.1}
$$

where

$$
\varrho_+(\chi_\omega) = \varrho_2 + \chi_\omega(x)(\varrho_1 - \varrho_2) \quad \text{and} \quad c_+(\chi_\omega) = c_2 + \chi_\omega(x)(c_1 - c_2),
$$

are the respective arithmetic means. We assume throughout that

$$
0 < c_1 < c_2. \tag{1.2}
$$

Through the separation of space and time variables one arrives at the temperature

$$
u(x,t) = \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda_k(c_+(\chi_\omega), \varrho_+(\chi_\omega))t} \langle u_0, \varrho_+(\chi_\omega) u_k \rangle u_k(x), \tag{1.3}
$$

where $\lambda_k(c_+(\chi_\omega), \varrho_+(\chi_\omega))$ is the kth eigenvalue and $u_k(x)$ is a kth eigenfunction of

$$
-\nabla \cdot c_+(\chi_\omega) \nabla v = \lambda \varrho_+(\chi_\omega) v, \quad v \in H_0^1(\Omega). \tag{1.4}
$$

From (1.3) it is clear that the long time behavior of u is determined by the eigenvalues. We aim then to expose the dependence of λ_k upon ω. In very special geometries, e.g., when ω and Ω are concentric balls, this dependence can be made explicit. In the general case one may at least hope to determine the numerical range of the λ_k. As such, we study the extremal eigenvalue problems

$$\inf_{\chi \in ad_\gamma} \lambda_k(c_+(\chi), \varrho_+(\chi)) \tag{1.5}$$

$$\sup_{\chi \in ad_\gamma} \lambda_k(c_+(\chi), \varrho_+(\chi)) \tag{1.6}$$

where

$$ad_\gamma \equiv \{\chi \in L^\infty(\Omega) : \chi(x) \in \{0,1\} \text{ a.e.}, \int_\Omega \chi \, dx = \gamma\}.$$

In the case of constant conductivity, i.e., $c_1 = c_2$, both (1.5) and (1.6) have been addressed by Tahraoui [T], Cox and McLaughlin [CM], and Cox [Co1]. In the case of constant heat capacity, i.e., $\varrho_1 = \varrho_2$, Alvino, Trombetti, and Lions [ATL], have considered (1.5) when Ω is a ball while Berryman [B] obtained upper bounds on λ_k over smooth conductivities. To our knowledge, the only previous work permitting variation in both c and ϱ is Jouron [J]. His techniques apply to (1.6) when $k = 1$.

It is well known that neither (1.5) nor (1.6) need possess a solution. In §2, following the work of Murat, Tartar, Lurie, and Cherkaev, especially [MT], [Lu] and [LC3], on the associated static conductivity problem, we establish the existence of solutions to relaxed formulations of (1.5) and (1.6). This amounts to extending ad_γ and λ_k to the associated class of extremal laminates. We remark that eigenvalues in fact yield pointwise bounds on solutions to the static problem. For example, if $w \in H_0^1(\Omega)$ satisfies $-\nabla \cdot c_+ \nabla w = \varrho_+$ then, as u_1, the associated first eigenfunction, is nonnegative,

$$\begin{aligned}\int_\Omega \varrho_+ u_1 \, dx &= -\int_\Omega u_1 \nabla \cdot c_+ \nabla w \, dx = -\int_\Omega w \nabla \cdot c_+ \nabla u_1 \, dx \\ &= \lambda_1(c_+, \varrho_+) \int_\Omega w \varrho_+ u_1 \, dx \le \lambda_1(c_+, \varrho_+) \|w\|_\infty \int_\Omega \varrho_+ u_1 \, dx.\end{aligned}$$

Hence, the maximum temperature satisfies

$$\|w\|_\infty \ge \inf_\chi \frac{1}{\lambda_1(c_+(\chi), \varrho_+(\chi))}.$$

In §§3 and 4 we derive necessary conditions, in terms of the generalized gradient of Clarke [Cl], for the extremizers of the relaxed problems. These necessary conditions constitute pointwise characterizations of the optimal laminar geometries. For the two dimensional case the conditions indicate that rank one laminates are necessary for simple eigenvalues and that rank two are necessary for the multiple eigenvalue case although in neither case have we been able to exhibit an explicit solution. In the one dimensional case, we show in §5 that the problem may be reduced to one of constant conductivity. This permits us, following Krein [Kr], to construct explicit solutions of the unrelaxed design problems. We find that the solutions bifurcate as one crosses the surface defined by $c_1 \varrho_1 = c_2 \varrho_2$. We close in §6 with comments on the extension of our theories to the case in which $c_1 = 0$ and, separately, to the case in which (1.1) is replaced by the linear system of elastodynamics.

2. Relaxation

One first permits simple mixtures by allowing weak$*$, with respect to $L^\infty(\Omega)$, limits of classical designs. In particular, one replaces ad_γ with

$$\theta \in ad^*_\gamma \equiv \{\vartheta \in L^\infty(\Omega) : 0 \le \vartheta(x) \le 1 \text{ a.e.}, \int_\Omega \vartheta\, dx = \gamma\}.$$

Jouron [J] has shown that this extension is in fact sufficient when considering (1.6) at $k = 1$. In particular, see [J, Corollary 2.2], he finds $\theta \mapsto \lambda_1(c_+(\theta), \varrho_+(\theta))$ to be weak$*$ upper semicontinuous. In order to accommodate both higher eigenvalues and minimization problems we are compelled to relax the dependence of c upon θ. The relaxation introduced for the problem of static conductivity will suffice. In particular, Given $\theta \in ad^*_\gamma$ one permits c to range over the class of matrix valued functions M_θ, see [MT, Proposition 10]. With

$$\mathcal{G} \equiv \{(\theta, c) : \theta \in ad^*_\gamma,\ c \in M_\theta\}$$

we may piece our first proposition from standard compactness and continuity results.

Proposition 2.1.

$$\inf_{\chi \in ad_\gamma} \lambda_k(c_+(\chi), \varrho_+(\chi)) = \min_{(\theta,c)\in\mathcal{G}} \lambda_k(c, \varrho_+(\theta))$$
$$\sup_{\chi \in ad_\gamma} \lambda_k(c_+(\chi), \varrho_+(\chi)) = \max_{(\theta,c)\in\mathcal{G}} \lambda_k(c, \varrho_+(\theta))$$

Proof: Suppose that $\{\chi_n\} \subset ad_\gamma$ is a minimizing sequence for λ_k. It follows that a subsequence of $\{\chi_n\}$ converges to $\theta \in ad^*_\gamma$ in the weak$*$ topology on $L^\infty(\Omega)$. As a result, $\varrho_+(\chi_n) \stackrel{*}{\rightharpoonup} \varrho_+(\theta)$ as well. In addition, see [MT, Proposition 8], $c_+(\chi_n)$ H–converges to a matrix $c^* \in M_\theta$ and conversely, every pair in $\mathcal{G}$ is constructed by such a limiting procedure. As $(c, \varrho) \mapsto \lambda_k(c, \varrho)$ is continuous with respect to H–convergence in c and weak$*$ convergence in ϱ, see, e.g., Boccardo and Marcellini [BM], it follows that λ_k attains its minimum over $\mathcal{G}$. ∎

Following Tartar [Tar] and Lurie and Cherkaev [LC2] we shall see, given θ, that it suffices to restrict c to one of the two forms

$$\overline{c}(\theta, T) = c_2 - \theta \left[\frac{1}{c_2 - c_1} - \frac{(1-\theta)T}{c_2}\right]^{-1} \tag{2.1}$$

$$\underline{c}(\theta, T) = c_1 + (1-\theta) \left[\frac{1}{c_2 - c_1} + \frac{\theta T}{c_1}\right]^{-1}. \tag{2.2}$$

where $T \in S^+_1$, the class of symmetric positive semidefinite unit trace matrices. For convenience, let us denote by

$$\mathcal{L} \equiv L^\infty(\Omega, S^+_1),$$

the bounded functions on Ω taking values in S_1^+. That $\overline{c}$ and $\underline{c}$ correspond to respectively efficient and poor conductors follows from

Proposition 2.2. *If $(\theta, c) \in \mathcal{G}$ then there exist $\underline{T}$ and $\overline{T}$ in $\mathcal{L}$ such that*

$$\underline{c}(\theta(x), \underline{T}(x)) \leq c(x) \leq \overline{c}(\theta(x), \overline{T}(x)) \tag{2.3}$$

in the sense of quadratic forms for almost every $x \in \Omega$.

Proof: The proof follows immediately upon noting that the Hashin-Shtrikman bounds for anisotropic composites can be written

$$\underline{c}(\theta, T) \leq c \leq \overline{c}(\theta, T) \tag{2.4}$$

for some T in $\mathcal{L}$. We remark that Hashin-Shtrikman bounds for anisotropic composites are established in the work of Kohn and Milton [KM]. ∎

To extend such inequalities to the eigenvalues of the conductor it suffices to recall the characterization

$$\lambda_k(c, \varrho) = \min_{H \in \mathcal{H}_k} \max_{u \in H} R(c, \varrho, u), \qquad R(c, \varrho, u) = \frac{\int_\Omega c \nabla u \cdot \nabla u \, dx}{\int_\Omega \varrho u^2 \, dx}. \tag{2.5}$$

Here $\mathcal{H}_k$ is the collection of k–dimensional subspaces of $H_0^1(\Omega)$.

Proposition 2.3.

$$\inf_{\chi \in ad_\gamma} \lambda_k(c_+(\chi), \varrho_+(\chi)) = \min_{\theta \in ad_\gamma^*} \min_{T \in \mathcal{L}} \lambda_k(\underline{c}(\theta, T), \varrho_+(\theta))$$
$$\sup_{\chi \in ad_\gamma} \lambda_k(c_+(\chi), \varrho_+(\chi)) = \max_{\theta \in ad_\gamma^*} \max_{T \in \mathcal{L}} \lambda_k(\underline{c}(\theta, T), \varrho_+(\theta)).$$

Proof: Suppose $(\check{\theta}_k, \check{c})$ minimizes $(\theta, c) \mapsto \lambda_k(c, \varrho_+(\theta))$ over $\mathcal{G}$. The previous Proposition now guarantees a $\underline{T} \in \mathcal{L}$ for which

$$R(\underline{c}(\check{\theta}_k, \underline{T}), \varrho_+(\check{\theta}_k), u) \leq R(\check{c}, \varrho_+(\check{\theta}_k), u) \quad \forall\, u \in H_0^1(\Omega).$$

From (2.5) it now follows that $\lambda_k(\underline{c}(\check{\theta}_k, \underline{T}), \varrho_+(\check{\theta}_k)) \leq \lambda_k(\check{c}, \varrho_+(\check{\theta}_k))$. As $\underline{c}(\check{\theta}_k, \underline{T}) \in M_{\check{\theta}_k}$ the first claim follows. For the second use $\overline{c}$ and $\overline{T}$ in place of $\underline{c}$ and $\underline{T}$. ∎

In the following, let us denote

$$\underline{\lambda}_k(\theta, T) \equiv \lambda_k(\underline{c}(\theta, T), \varrho_+(\theta)) \quad \text{and} \quad \overline{\lambda}_k(\theta, T) \equiv \lambda_k(\overline{c}(\theta, T), \varrho_+(\theta)).$$

3. Minimizing $\underline{\lambda}_k$

We view $\underline{\lambda}_k$ as a function on $L^\infty(\Omega) \times L^\infty(\Omega, S)$ in the strong product topology where S denotes the class of symmetric matrices. We denote its minimum over $ad_\gamma^* \times \mathcal{L}$ by $\check{\lambda}_k$.

If the pair $(\check{\theta}_k, \check{T}_k)$ minimizes $\underline{\lambda}_k$ over $ad^*_\gamma \times \mathcal{L}$ and $\underline{\lambda}_k$ is Lipschitz near $(\check{\theta}_k, \check{T}_k)$ then necessarily, see, e.g., Clarke [Cl, 2.4.3 Corollary],

$$0 \in \partial\underline{\lambda}_k(\check{\theta}_k, \check{T}_k) + N_{ad^*_\gamma \times \mathcal{L}}(\check{\theta}_k, \check{T}_k), \tag{3.1}$$

where $\partial\underline{\lambda}_k$ denotes the generalized gradient of $\underline{\lambda}_k$ and $N_{ad^*_\gamma \times \mathcal{L}}$ is the cone of normals to $ad^*_\gamma \times \mathcal{L}$. Our calculation of these two objects will produce pointwise characterizations of $\check{\theta}_k$ and $\check{T}_k$ in terms of the eigenfunctions associated with $\underline{\lambda}_k(\check{\theta}_k, \check{T}_k)$.

Proposition 3.1. *$\underline{\lambda}_k$ is Lipschitz in a neighborhood of $(\check{\theta}_k, \check{T}_k)$ and its generalized gradient at this point is*

$$\partial\underline{\lambda}_k(\check{\theta}_k, \check{T}_k) = \mathrm{co}\,\{\partial\underline{c}(\check{\theta}_k, \check{T}_k)\nabla u \cdot \nabla u - \check{\lambda}_k \partial\varrho_+(\check{\theta}_k)u^2 : u \in E^1_k(\check{\theta}_k, \check{T}_k)\}$$

where co denotes convex hull and $E^1_k(\check{\theta}_k, \check{T}_k)$ is the intersection of the eigenspace corresponding to $\check{\lambda}_k$ and the unit sphere in $L^2(\Omega)$. The respective gradients of $\underline{c}$ and ϱ_+ are

$$\langle \partial\underline{c}(\check{\theta}_k, \check{T}_k), [\theta, T]\rangle = -\theta\check{B}_k\left[I + \frac{1-\check{\theta}_k}{c_1}\check{T}_k\check{B}_k\right] - \frac{\check{\theta}_k(1-\check{\theta}_k)}{c_1}\check{B}_k T \check{B}_k,$$
$$\text{where} \quad \check{B}_k = \left[\frac{I}{c_2 - c_1} + \frac{\check{\theta}_k\check{T}_k}{c_1}\right]^{-1},$$

and

$$\langle \partial\varrho_+(\check{\theta}_k), [\theta, T]\rangle = (\varrho_1 - \varrho_2)\theta.$$

Proof: This calculation follows from Cox [Co2] upon observing that $A(\theta, T) \equiv -\nabla\cdot\underline{c}(\theta, T)\nabla$ is a C^1 mapping from a neighborhood of $(\check{\theta}_k, \check{T}_k)$ into the symmetric positive isomorphisms of $H^1_0(\Omega)$ and its dual and that multiplication by $\varrho_+(\theta)$ is a symmetric, positive, isomorphism of $L^2(\Omega)$. ∎

As the normal cone of a product is the product of the normal cones, see [Cl, 2.4.5 Corollary], we find that (3.1) is equivalent to the statement: There exists a pair $(\check{\xi}_k, \check{\Xi}_k) \in \partial\underline{\lambda}_k(\check{\theta}_k, \check{T}_k)$ such that $-\check{\xi}_k \in N_{ad^*_\gamma}(\check{\theta}_k)$ and $-\check{\Xi}_k \in N_{\mathcal{L}}(\check{T}_k)$. As both ad^*_γ and $\mathcal{L}$ are convex it follows, [Cl, 2.4.4 Proposition], that these two inclusions are equivalent to

$$\int_\Omega \langle \check{\xi}_k, \check{\theta}_k\rangle\, dx = \min_{\theta \in ad^*_\gamma} \int_\Omega \langle \check{\xi}_k, \theta\rangle\, dx \tag{3.2}$$

$$\int_\Omega \langle \check{\Xi}_k, \check{T}_k\rangle\, dx = \min_{T \in \mathcal{L}} \int_\Omega \langle \check{\Xi}_k, T\rangle\, dx. \tag{3.3}$$

We first address the latter. If $(\check{\xi}_k, \check{\Xi}_k) \in \partial\underline{\lambda}_k(\check{\theta}_k, \check{T}_k)$ then

$$\langle \check{\Xi}_k, T\rangle = -\frac{(1-\check{\theta}_k)\check{\theta}_k}{c_1}\sum_{i=1}^p \mu_i \langle T\check{B}_k\nabla u_i, \check{B}_k\nabla u_i\rangle = -\frac{(1-\check{\theta}_k)\check{\theta}_k}{c_1}\mathrm{tr}\,(T\check{F}_k),$$

where $u_i \in E_k^1(\check{\theta}_k, \check{T}_k)$, $\mu_i \geq 0$, $\sum_{i=1}^{p} \mu_i = 1$, and

$$\check{F}_k = \sum_{i=1}^{p} \mu_i \check{B}_k \nabla u_i \otimes \check{B}_k \nabla u_i. \tag{3.4}$$

We now deduce proerties of (3.3) as the pointwise minimizer of $T \mapsto \langle \check{\Xi}_k, T \rangle$ at each point in Ω. For algebraic convenience we carry out this, and the remaining, calculations under the assumption that Ω lies in the plane, i.e., $d = 2$. In this case the spectral decomposition of $\check{F}_k$ is

$$\check{F}_k = f_1 n \otimes n + f_2 n^\perp \otimes n^\perp, \quad f_1 \leq f_2.$$

We show that $\check{T}_k$ shares this decomposition and determine n in terms of the ∇u_i.

Proposition 3.2. *The matrix $\check{T}_k$ takes the form*

$$\check{T}_k = \check{t}_1 n \otimes n + \check{t}_2 n^\perp \otimes n^\perp$$

where $\check{t}_1$ and $\check{t}_2$ are nonnegative and sum to one and the direction n is determined by

$$\sum_{i=1}^{p} \mu_i (n \cdot \nabla u_i)(n^\perp \cdot \nabla u_i) = 0.$$

Proof: If $0 \leq t_1 \leq t_2$ denote the eigenvalues of $T \in S_1^+$ then Mirsky's Theorem [M] gives

$$f_2 t_1 + f_1 t_2 \leq \operatorname{tr}(T \check{F}_k) \leq f_1 t_1 + f_2 t_2. \tag{3.5}$$

As $t_1 + t_2 = 1$ it follows that $\operatorname{tr}(T\check{F}_k) \leq f_2$. Equality is attained when

$$\check{T}_k = \begin{cases} n^\perp \otimes n^\perp, & \text{if} \quad f_1 < f_2, \\ \check{t}_1 n \otimes n + \check{t}_2 n^\perp \otimes n^\perp, & \text{if} \quad f_1 = f_2, \end{cases}$$

where $\check{t}_1$ and $\check{t}_2$ are yet to be determined. Upon writing $I = n \otimes n + n^\perp \otimes n^\perp$ we may now express $\check{B}_k$ as

$$\check{B}_k = b_1 n \otimes n + b_2 n^\perp \otimes n^\perp, \quad \text{where} \quad b_i = \frac{c_1(c_2 - c_1)}{c_1 + \check{\theta}_k (c_2 - c_1) \check{t}_i}.$$

Substituting this into (3.4) we find

$$\begin{aligned} f_1 n \otimes n + f_2 n^\perp \otimes n^\perp = b_1^2 n \otimes n \sum_{i=1}^{p} \mu_i (n \cdot \nabla u_i)^2 + b_2^2 n^\perp \otimes n^\perp \sum_{i=1}^{p} \mu_i (n^\perp \cdot \nabla u_i)^2 \\ + b_1 b_2 (n \otimes n^\perp + n^\perp \otimes n) \sum_{i=1}^{p} \mu_i (n \cdot \nabla u_i)(n^\perp \cdot \nabla u_i). \end{aligned} \tag{3.6}$$

and so

$$b_1^2 \sum_{i=1}^p \mu_i (n \cdot \nabla u_i)^2 = f_1 \quad \text{and} \quad b_2^2 \sum_{i=1}^p \mu_i (n^\perp \cdot \nabla u_i)^2 = f_2 \tag{3.7}$$

while

$$\sum_{i=1}^p \mu_i (n \cdot \nabla u_i)(n^\perp \cdot \nabla u_i) = 0. \tag{3.8}$$

To see that (3.8) indeed determines n we write

$$\operatorname{tr}(n \otimes n^\perp \check{H}_k) = \sum_{i=1}^p \mu_i (n \cdot \nabla u_i)(n^\perp \cdot \nabla u_i) \quad \text{where} \quad \check{H}_k = \sum_{i=1}^p \mu_i \nabla u_i \otimes \nabla u_i.$$

With regard to the spectral decomposition

$$\check{H}_k = h_1 v \otimes v + h_2 v^\perp \otimes v^\perp,$$

it is easily seen for the choice $n = (v + v^\perp)/\sqrt{2}$ and $n^\perp = (v - v^\perp)/\sqrt{2}$ that

$$\operatorname{tr}(n \otimes n^\perp \check{H}_k) = (h_1 - h_2)/2,$$

where as for $n = (v^\perp - v)/\sqrt{2}$ and $n^\perp = (v^\perp + v)/\sqrt{2}$ that

$$\operatorname{tr}(n \otimes n^\perp \check{H}_k) = -(h_1 - h_2)/2.$$

It now follows from the intermediate value theorem that (3.8) has a solution.

Regarding $\check{t}_1$ and $\check{t}_2$ we noted above that where $f_1 < f_2$ we may simply choose $\check{t}_1 = 0$ and $\check{t}_2 = 1$. When $f_1 = f_2$ we find that (3.7) together with $\check{t}_1 + \check{t}_2 = 1$ determines $\check{t}_1$ and $\check{t}_2$. ∎

We remark that if $\check{\lambda}_k$ is simple then the u_i coincide. We naturally label this function $\check{u}_k$ and note that this implies that $\check{F}_k$ is rank one. In this case $f_1 = 0$ and so (3.7) states $n \cdot \nabla \check{u}_k = 0$. Hence,

Corollary 3.3. *If $\underline{\lambda}_k(\check{\theta}_k, \check{T}_k)$ is simple then* $\check{T}_k = \dfrac{\nabla \check{u}_k \otimes \nabla \check{u}_k}{|\nabla \check{u}_k|^2}$.

On inserting this into $\underline{c}$ we find in the simple case that $\check{T}_k$ selects the harmonic mean. That is,

$$\underline{c}(\check{\theta}_k, \check{T}_k)\nabla \check{u}_k = \frac{c_1 c_2}{c_1 + \check{\theta}_k (c_2 - c_1)} \nabla \check{u}_k = c_-(\check{\theta}_k) \nabla \check{u}_k \tag{3.9}$$

where $c_-(\check{\theta}_k)$ is the harmonic mean of c_1 and c_2 in volume fraction $\check{\theta}_k$.

We now take up condition (3.2) in our characterization of $\check{\theta}_k$. From Proposition 3.1 it follows that

$$\begin{aligned}
\check{\xi}_k &= -\sum_{i=1}^p \mu_i \left\{ \langle \check{B}_k \nabla u_i, \nabla u_i \rangle + \frac{1 - \check{\theta}_k}{c_1} \langle \check{T}_k \check{B}_k \nabla u_i, \check{B}_k \nabla u_i \rangle - \check{\lambda}_k (\varrho_2 - \varrho_1) u_i^2 \right\} \\
&= -(c_2 - c_1) \sum_{i=1}^p \mu_i \left\{ \langle (I - \check{T}_k) \nabla u_i, \nabla u_i \rangle + \frac{c_1 c_2 \langle \check{T}_k \nabla u_i, \nabla u_i \rangle}{(c_1 + \check{\theta}_k (c_2 - c_1))^2} - \check{\lambda}_k \frac{\varrho_2 - \varrho_1}{c_2 - c_1} u_i^2 \right\}.
\end{aligned}$$

We remove the volume constraint in ad_γ^* at the cost of a Lagrange multiplier. In particular, from the Lagrange Multiplier Rule, [Cl, Theorem 6.1.1], we deduce that (3.2) implies that

$$\int_\Omega (\nu_1 \check{\xi}_k + \nu_2)\check{\theta}_k \, dx = \min_{\theta \in ad^*} \int_\Omega (\nu_1 \check{\xi}_k + \nu_2)\theta \, dx.$$

where $\nu_1 \geq 0$ and $|\nu_1| + |\nu_2| > 0$. Now, if $\nu_1 = 0$ then $\nu_2 \neq 0$ and $\check{\theta}_k$ must be either identically zero or one, in violation of the volume constraint. With $\nu = -\nu_2/\nu_1$ we find

$$\check{\theta}_k = 0 \Rightarrow \check{\xi}_k \geq \nu, \tag{3.10}$$

$$0 < \check{\theta}_k < 1 \Rightarrow \check{\xi}_k = \nu \tag{3.11}$$

$$\check{\theta}_k = 1 \Rightarrow \check{\xi}_k \leq \nu. \tag{3.12}$$

The import of the these conditions become much more visible when $\check{\lambda}_k$ is simple. For in this case

$$\check{\xi}_k = -\frac{c_1 c_2 (c_2 - c_1)}{(c_1 + \check{\theta}_k (c_2 - c_1))^2} |\nabla \check{u}_k|^2 - \check{\lambda}_k (\varrho_1 - \varrho_2) \check{u}_k^2$$

in which case (3.10)–(3.12) take the form

$$\check{\theta}_k = 0 \Rightarrow c_-^2(\check{\theta}_k)|\nabla \check{u}_k|^2 \leq c_1 c_2 f_\nu(\check{\lambda}_k, \check{u}_k), \tag{3.13}$$

$$0 < \check{\theta}_k < 1 \Rightarrow c_-^2(\check{\theta}_k)|\nabla \check{u}_k|^2 = c_1 c_2 f_\nu(\check{\lambda}_k, \check{u}_k), \tag{3.14}$$

$$\check{\theta}_k = 1 \Rightarrow c_-^2(\check{\theta}_k)|\nabla \check{u}_k|^2 \geq c_1 c_2 f_\nu(\check{\lambda}_k, \check{u}_k), \tag{3.15}$$

where c_- denotes the harmonic mean, see (3.9), and

$$f_\nu(\lambda, u) = \frac{\nu + \lambda(\varrho_2 - \varrho_1)u^2}{c_2 - c_1}. \tag{3.16}$$

Conditions (3.13)–(3.15) are similar in form to those obtained by Lurie and Cherkaev [LC1] in their study of torsional rigidity and to those obtained by Murat and Tartar [MT, Eq.(60)] in their maximization of the average temperature in a statically loaded conductor. Remark 39 of Murat and Tartar remains true in our context.

Proposition 3.4. *If $\check{\lambda}_k$ is simple and $\check{\theta}_k$ changes discontinuously from 0 to 1 across a smooth curve Γ then $\nabla \check{u}_k$ is perpendicular to Γ.*

Proof: Suppose at $0 \in \Gamma$ that $(0,1)$ is normal to Γ. As

$$-\nabla \cdot c_-(\check{\theta}_k)\nabla \check{u}_k = \check{\lambda}_k \varrho_+(\check{\theta}_k)\check{u}_k$$

it follows that $\check{u}_k$ and $c_-(\check{\theta}_k)\partial_{x_2}\check{u}_k$ are continuous across Γ at 0. We denote by 0_- and 0_+ limits as one approaches 0 in regions where $\check{\theta}_k = 0$ and $\check{\theta}_k = 1$ respectively. From (3.13) we find

$$c_2^2(\partial_{x_1}\check{u}_k)^2(0) \leq c_1 c_2 f_\nu(\check{\lambda}_k, \check{u}_k(0)) - c_2^2(\partial_{x_2}\check{u}_k)^2(0_-), \tag{3.17}$$

while (3.15) implies

$$c_1 c_2 f_\nu(\check{\lambda}_k, \check{u}_k(0)) - c_1^2(\partial_{x_2}\check{u}_k)^2(0_+) \le c_1^2(\partial_{x_1}\check{u}_k)^2(0). \tag{3.18}$$

Combining (3.17) and (3.18) we find $\partial_{x_1}\check{u}_k(0) = 0$ as claimed. ∎

4. **Maximizing** $\overline{\lambda}_k$

We apply the tools of the previous section to the characterization of $(\hat{\theta}_k, \hat{T}_k)$, a maximizer of $\overline{\lambda}_k$ over $ad^*_\gamma \times \mathcal{L}$. The necessary condition in this case is

$$0 \in -\partial\overline{\lambda}_k(\hat{\theta}_k, \hat{T}_k) + N_{ad^*_\gamma \times \mathcal{L}}(\hat{\theta}_k, \hat{T}_k),$$

where the generalized gradient of $\overline{\lambda}_k$ at $(\hat{\theta}_k, \hat{T}_k)$ is

$$\partial\overline{\lambda}_k(\hat{\theta}_k, \hat{T}_k) = \mathrm{co}\,\{\partial\overline{c}(\hat{\theta}_k, \hat{T}_k)\nabla v \cdot \nabla v - \hat{\lambda}_k \partial\varrho_+(\hat{\theta}_k)v^2 : v \in E_k^1(\hat{\theta}_k, \hat{T}_k)\}$$

where $\hat{\lambda}_k$ is now short for $\overline{\lambda}_k(\hat{\theta}_k, \hat{T}_k)$ and $E_k^1(\hat{\theta}_k, \hat{T}_k)$ is the intersection of the eigenspace corresponding to $\hat{\lambda}_k$ and the unit sphere in $L^2(\Omega)$. The gradient of $\overline{c}$ is

$$\langle \partial\overline{c}(\hat{\theta}_k, \hat{T}_k), (\theta, T)\rangle = \frac{\theta}{c_2}\hat{B}_k(\hat{\theta}_k\hat{T}_k\hat{B}_k - c_2 I) - \frac{(1-\hat{\theta}_k)\hat{\theta}_k}{c_2}\hat{B}_k T \hat{B}_k$$

where

$$\hat{B}_k = \left[\frac{I}{c_2 - c_1} - \frac{(1-\hat{\theta}_k)\hat{T}_k}{c_2}\right]^{-1}.$$

Analogous to (3.2) and (3.3) we conclude that $\partial\overline{\lambda}_k(\hat{\theta}_k, \hat{T}_k)$ contains a pair $(\hat{\xi}_k, \hat{\Xi}_k)$ that satisfies the maximum problems

$$\int_\Omega \langle \hat{\xi}_k, \hat{\theta}_k\rangle\, dx = \max_{\theta \in ad^*_\gamma} \int_\Omega \langle \hat{\xi}_k, \theta\rangle\, dx, \tag{4.1}$$

$$\int_\Omega \langle \hat{\Xi}_k, \hat{T}_k\rangle\, dx = \max_{T \in \mathcal{L}} \int_\Omega \langle \hat{\Xi}_k, T\rangle\, dx. \tag{4.2}$$

Addressing first the latter we find

$$\langle \hat{\Xi}_k, T\rangle = -\frac{(1-\hat{\theta}_k)\hat{\theta}_k}{c_2}\sum_{i=1}^m \tau_i \langle T\hat{B}_k\nabla v_i, \hat{B}_k\nabla v_i\rangle = -\frac{(1-\hat{\theta}_k)\hat{\theta}_k}{c_2}\mathrm{tr}\,(T\hat{F}_k),$$

where $v_i \in E_k^1(\hat{\theta}_k, \hat{T}_k)$, $\tau_i \ge 0$, $\sum_{i=1}^m \tau_i = 1$, and

$$\hat{F}_k = \sum_{i=1}^m \tau_i \hat{B}_k\nabla v_i \otimes \hat{B}_k\nabla v_i. \tag{4.3}$$

Again confining our attention to the planar case, with

$$\hat{F}_k = f_1 n \otimes n + f_2 n^\perp \otimes n^\perp, \quad f_1 \le f_2$$

we establish the analog of Proposition 3.2.

Proposition 4.1. *The matrix $\hat{T}_k$ takes the form*

$$\hat{T}_k = \hat{t}_1 n \otimes n + \hat{t}_2 n^\perp \otimes n^\perp$$

where $\hat{t}_1$ and $\hat{t}_2$ are nonnegative and sum to one and the direction n is determined by

$$\sum_{i=1}^m \tau_i (n \cdot \nabla v_i)(n^\perp \cdot \nabla v_i) = 0.$$

Proof: From the left side of (3.5) we see that $f_1 \le \operatorname{tr}(T\hat{F}_k)$ for each $T \in S_1^+$. Equality is attained when

$$\hat{T}_k = \begin{cases} n \otimes n, & \text{if} \quad f_1 < f_2, \\ \hat{t}_1 n \otimes n + \hat{t}_2 n^\perp \otimes n^\perp, & \text{if} \quad f_1 = f_2, \end{cases}$$

where $\hat{t}_1$ and $\hat{t}_2$ are yet to be determined. Upon writing $I = n \otimes n + n^\perp \otimes n^\perp$ we may express $\hat{B}_k$ as

$$\hat{B}_k = b_1 n \otimes n + b_2 n^\perp \otimes n^\perp \quad \text{where} \quad b_i = \frac{c_2(c_2 - c_1)}{c_2 - (1 - \hat{\theta}_k)(c_2 - c_1)\hat{t}_i}.$$

Substituting this into (4.3) we find

$$\begin{aligned} f_1 n \otimes n + f_2 n^\perp \otimes n^\perp = b_1^2 n \otimes n \sum_{i=1}^m \tau_i (n \cdot \nabla v_i)^2 + b_2^2 n^\perp \otimes n^\perp \sum_{i=1}^m \tau_i (n^\perp \cdot \nabla v_i)^2 \\ + b_1 b_2 (n \otimes n^\perp + n^\perp \otimes n) \sum_{i=1}^m \tau_i (n \cdot \nabla v_i)(n^\perp \cdot \nabla v_i). \end{aligned} \tag{4.4}$$

and so

$$b_1^2 \sum_{i=1}^m \tau_i (n \cdot \nabla v_i)^2 = f_1 \quad \text{and} \quad b_2^2 \sum_{i=1}^m \tau_i (n^\perp \cdot \nabla v_i)^2 = f_2 \tag{4.5}$$

while

$$\sum_{i=1}^m \tau_i (n \cdot \nabla v_i)(n^\perp \cdot \nabla v_i) = 0.$$

As argued in the previous section, such an equation determines the direction n.

Regarding $\hat{t}_1$ and $\hat{t}_2$, if $f_1 < f_2$ then $\hat{t}_1 = 1$ and $\hat{t}_2 = 0$, while, if $f_1 = f_2$ then one must invoke (4.5) and $\hat{t}_1 + \hat{t}_2 = 1$. ∎

We remark that if $\hat{\lambda}_k$ is simple then the v_i coincide. We call this function $\hat{v}_k$ and note, in this case, that $\hat{F}_k$ is rank one. Hence $f_1 = 0$ and (4.5) requires $n \cdot \nabla \hat{v}_k = 0$. As a result,

Corollary 4.2. *If $\hat{\lambda}_k$ is simple then* $\hat{T}_k = \dfrac{\nabla \hat{v}_k^{\perp} \otimes \nabla \hat{v}_k^{\perp}}{|\nabla \hat{v}_k^{\perp}|^2}$.

On inserting this into $\overline{c}$ we find that $\hat{T}_k$ selects the arithmetic, rather than the harmonic, mean

$$\overline{c}(\hat{\theta}_k, \hat{T}_k)\nabla \hat{v}_k = (c_2 - \hat{\theta}_k(c_2 - c_1))\nabla \hat{v}_k = c_+(\hat{\theta}_k)\nabla \hat{v}_k.$$

We now turn to the characterization of $\hat{\theta}_k$. From Proposition 4.1 it follows that

$$\begin{aligned}\hat{\xi}_k &= -\sum_{i=1}^m \tau_i \left\{ \langle \hat{B}_k \nabla v_i, \nabla v_i \rangle - \frac{\hat{\theta}_k}{c_2} \langle \hat{T}_k \hat{B}_k \nabla v_i, \hat{B}_k \nabla v_i \rangle - \hat{\lambda}_k(\varrho_2 - \varrho_) v_i^2 \right\} \\ &= -(c_2 - c_1)\sum_{i=1}^m \tau_i \left\{ |\nabla v_i|^2 + \eta \langle \hat{T}_k \nabla v_i, \nabla v_i \rangle - \hat{\lambda}_k \frac{\varrho_2 - \varrho_1}{c_2 - c_1} v_i^2 \right\}.\end{aligned}$$

where

$$\eta = (c_2 - c_1)(1 - \hat{\theta}_k) - \frac{c_2 \hat{\theta}_k}{(c_1 + \hat{\theta}_k(c_2 - c_1))^2}.$$

Given the previous Corollary we find in the simple case that

$$\hat{\xi}_k = -(c_2 - c_1)|\nabla \hat{v}_k|^2 + \hat{\lambda}_k(\varrho_2 - \varrho_1)\hat{v}_k^2. \tag{4.6}$$

Invoking the Lagrange Multiplier Rule to accommodate the volume constraint we find that $\hat{\theta}_k$ and $\hat{\xi}_k$ are coupled through

$$\begin{aligned}\hat{\theta}_k = 0 &\Rightarrow \hat{\xi}_k \le \nu \\ 0 < \hat{\theta}_k < 1 &\Rightarrow \hat{\xi}_k = \nu \\ \hat{\theta}_k = 1 &\Rightarrow \hat{\xi}_k \ge \nu\end{aligned}$$

When $\hat{\lambda}_k$ is simple, given (4.6), this takes the more manageable form

$$\hat{\theta}_k = 0 \Rightarrow |\nabla \hat{v}_k|^2 \ge f_\nu(\hat{\lambda}_k, \hat{v}_k) \tag{4.7}$$
$$0 < \hat{\theta}_k < 1 \Rightarrow |\nabla \hat{v}_k|^2 = f_\nu(\hat{\lambda}_k, \hat{v}_k) \tag{4.8}$$
$$\hat{\theta}_k = 1 \Rightarrow |\nabla \hat{v}_k|^2 \le f_\nu(\hat{\lambda}_k, \hat{v}_k), \tag{4.9}$$

where f_ν is still given by (3.16). These conditions are analogous to those obtained by Murat and Tartar [MT, Eq.(80)] in their minimization of the average temperature in a statically loaded conductor and to those obtained by Lurie and Cherkaev in their study of torsional rigidity [LC1]. We check the converse to Proposition 3.4.

Proposition 4.3. *If $\hat{\lambda}_k$ is simple and $\hat{\theta}_k$ changes discontinuously from 0 to 1 across a smooth curve Γ then $\nabla \hat{v}_k$ is parallel to Γ.*

Proof: Suppose at $0 \in \Gamma$ and that $(0,1)$ is normal to Γ. As

$$-\nabla \cdot c_+(\hat{\theta}_k)\nabla \hat{v}_k = \hat{\lambda}_k \varrho_+(\hat{\theta}_k)\hat{v}_k$$

we find $\hat{v}_k$ and $c_+(\hat{\theta}_k)\partial_{x_2}\hat{v}_k$ to be continuous across Γ at 0. We denote by 0_- and 0_+ limits as one approaches 0 in regions where $\hat{\theta}_k = 0$ and $\hat{\theta}_k = 1$ respectively. Continuity of $c_+(\hat{\theta}_k)\partial_{x_2}\hat{v}_k$ requires, in particular, that

$$c_2\partial_{x_2}\hat{v}_k(0_-) = c_1\partial_{x_2}\hat{v}_k(0_+). \tag{4.10}$$

From (4.7) we find

$$(\partial_{x_2}\hat{v}_k)^2(0_-) \geq f_\nu(\hat{\lambda}_k, \hat{v}_k(0)) - (\partial_{x_1}\hat{v}_k)^2(0), \tag{4.11}$$

while (4.9) implies

$$(\partial_{x_2}\hat{v}_k)^2(0_+) \leq f_\nu(\hat{\lambda}_k, \hat{v}_k(0)) - (\partial_{x_1}\hat{v}_k)^2(0). \tag{4.12}$$

Combining (3.17) and (3.18) we find $|\partial_{x_2}\hat{v}_k(0_+)| \leq |\partial_{x_2}\hat{v}_k(0_-)|$. As $c_1 < c_2$ however this contradicts (4.10) unless $\partial_{x_2}\hat{v}_k(0_+) = \partial_{x_2}\hat{v}_k(0_-) = 0$. ∎

5. The One–Dimensional Case

Via a simple change of variable we lump the conductivity with the lower order term. This reduces our extremization problems to ones treated by Krein [Kr]. We produce explicit characterizations of these extremizers and show that minimizers become maximizers, and conversely, upon crossing the the surface defined by $c_1\varrho_1 = c_2\varrho_2$.

When $d = 1$ the class $\mathcal{L}$ defaults to the singleton $T(x) \equiv 1$ and so

$$\underline{c}(\theta, T) = \overline{c}(\theta, T) = c_-(\theta) = \frac{c_1c_2}{c_1 + \theta(c_2 - c_1)}.$$

We therefore extremize $\lambda_k(\theta)$, the kth eigenvalue of

$$-(c_-(\theta)u')' = \lambda\varrho_+(\theta)u, \quad u(0) = u(1) = 0, \tag{5.1}$$

over $\theta \in ad^*_\gamma$. With respect to the new independent variable

$$y \equiv \int_0^x \frac{ds}{c_-(\theta(s))} = \frac{1}{c_1c_2}\int_0^x (c_1 + \theta(s)(c_2 - c_1))\,ds,$$

the unit interval is transformed to one of length

$$\ell \equiv y(1) = \frac{c_1 + \gamma(c_2 - c_1)}{c_1c_2},$$

and, upon setting $U(y) = u(x)$, where u satisfies (5.1), we find that U satisfies

$$-U''(y) = \lambda \Upsilon(y) U(y), \quad U(0) = U(\ell) = 0, \tag{5.2}$$

where

$$\Upsilon(y) = c_-(x)\varrho_+(x) = c_1 c_2 \frac{\varrho_2 + \theta(x)(\varrho_1 - \varrho_2)}{c_1 + \theta(x)(c_2 - c_1)}. \tag{5.3}$$

Regarding pointwise bounds on $\Upsilon(y)$ we note if

$$f(t) \equiv \frac{\varrho_2 + t(\varrho_1 - \varrho_2)}{c_1 + t(c_2 - c_1)} \quad \text{then} \quad f'(t) = \frac{c_1\varrho_1 - c_2\varrho_2}{(c_1 + t(c_2 - c_1))^2}.$$

As a result, $\Upsilon(y)$ attains its extremes at the extremes of θ. In particular

$$\min\{c_1\varrho_1, c_2\varrho_2\} \le \Upsilon(y) \le \max\{c_1\varrho_1, c_2\varrho_2\}. \tag{5.4}$$

for almost every $y \in (0,\ell)$. We remark that if $c_1\varrho_1 = c_2\varrho_2$ then Υ is *constant* and as such the eigenvalues of (5.1) are *independent* of θ. We note when $c_1\varrho_1 = c_2\varrho_2$ that the conductivity, $c_-(\theta)$, is inversely proportional to the heat capacity, $\varrho_+(\theta)$. In this case the eigenvalues are sensitive to the material properties and volume fraction but not to their relative distribution. In particular,

Proposition 5.1. *If* $c_1\varrho_1 = c_2\varrho_2$ *then*

$$\lambda_k(\theta) = \frac{k^2\pi^2}{c_1\varrho_1\ell^2} = \frac{c_1(k\pi c_2)^2}{\varrho_1(c_1 + \gamma(c_2 - c_1))^2}, \qquad \forall\,\theta \in ad_\gamma^*.$$

Let us now consider the case when $c_1\varrho_1 \ne c_2\varrho_2$. Integration of (5.3) yields

$$\int_0^\ell \Upsilon\,dy = \int_0^1 \varrho_+(\theta(x))\,dx = \varrho_2 + \gamma(\varrho_1 - \varrho_2). \tag{5.5}$$

We now recall that Krein [Kr] addresses the extreme values of each eigenvalue of (5.2) under the assumption that Υ is of fixed mean and is bounded uniformly from above and below. In our context, [Kr, Theorem 5] takes the form

Proposition 5.2. *Let* $D = \gamma/c_1$ *and suppose that* $c_1\varrho_1 < c_2\varrho_2$. *The function* $\Upsilon \mapsto \lambda_1(\Upsilon)$ *attains its unique minimum over those* Υ *satisfying* (5.4) *and* (5.5) *at*

$$\check{\Upsilon}(y) = \begin{cases} c_1\varrho_1 & \textit{if} \quad 0 < x < D/2, \\ c_2\varrho_2 & \textit{if} \quad D/2 < x < \ell - D/2, \\ c_1\varrho_1 & \textit{if} \quad \ell - D/2 < x < \ell, \end{cases} \tag{5.6}$$

and its unique maximum over the same set at

$$\hat{\Upsilon}(y) = \begin{cases} c_2\varrho_2 & \textit{if} \quad 0 < x < (\ell - D)/2, \\ c_1\varrho_1 & \textit{if} \quad (\ell - D)/2 < x < (\ell + D)/2, \\ c_2\varrho_2 & \textit{if} \quad (\ell + D)/2 < x < \ell. \end{cases} \tag{5.7}$$

If instead $c_1\varrho_1 > c_2\varrho_2$ then $\check{\Upsilon}$ is the maximizer and $\hat{\Upsilon}$ is the minimizer.

As the kth eigenvalue is the first eigenvalue of each of its nodal domains, Krein obtains the extremizers of λ_k by simply taking periodic extensions of $\check{\Upsilon}$ and $\hat{\Upsilon}$. It remains only for us to transform these two functions back to our original variables. From (5.3) and (5.6) comes

$$\check{\Upsilon}(y) = c_2\varrho_2 + \check{\theta}(x)(c_1\varrho_1 - c_2\varrho_2) \quad \text{and} \quad \check{\theta}(x) = \begin{cases} 0 & \text{if} \quad \gamma/2 < x < 1-\gamma/2, \\ 1 & \text{otherwise.} \end{cases}$$

In a like manner, (5.3) and (5.7) produce

$$\hat{\Upsilon}(y) = c_2\varrho_2 + \hat{\theta}(x)(c_1\varrho_1 - c_2\varrho_2) \quad \text{and} \quad \hat{\theta}(x) = \begin{cases} 1 & \text{if} \quad (1-\gamma)/2 < x < (1+\gamma)/2, \\ 0 & \text{otherwise.} \end{cases}$$

Taking the periodic extensions, in the sense of [Kr, Theorem 5], of $\check{\theta}$ and $\hat{\theta}$ we arrive at our final result.

Proposition 5.3. *If $c_1\varrho_1 < c_2\varrho_2$ then $\theta \mapsto \lambda_k(\theta)$*
(i) achieves its unique minimum over ad^*_γ at

$$\check{\theta}_k(x) = \begin{cases} 0 & \text{if} \quad \frac{2(j-1)+\gamma}{2k} \le x \le \frac{2j-\gamma}{2k} \quad j=1,\ldots,k; \\ 1 & \text{otherwise,} \end{cases} \qquad \text{and}$$

(ii) achieves its unique maximum over ad^*_γ at

$$\hat{\theta}_k(x) = \begin{cases} 1 & \text{if} \quad \frac{2j-1-\gamma}{2k} \le x \le \frac{2j-1+\gamma}{2k} \quad j=1,\ldots,k; \\ 0 & \text{otherwise.} \end{cases}$$

If instead $c_1\varrho_1 > c_2\varrho_2$ then $\check{\theta}_k$ is the maximizer and $\hat{\theta}_k$ is the minimizer.

6. **Comments**

We point to work relevant to the situations in which (i) the poor conductor approaches an insulator, and (ii) the problem of conductivity is replaced by one of elasticity.

When the conductivity of material one approaches zero we arrive at the shape problem corresponding to the optimal distribution of a single conductor within a prescribed region Ω. In this case one must address the nature of the boundary conditions to be placed on $\partial\omega \setminus \partial\Omega$, that part of the conductor's boundary that does not intersect the boundary of Ω. Imposing Dirichlet conditions on $\partial\omega$ we recall that Buttazzo and Dal Maso [BD] have constructed a successful relaxation of the minimization problem (1.5). In particular, they have shown that $Q \mapsto \lambda_k(Q)$ attains its minimum over those quasi–open subsets of Ω of prescribed volume. Here, $\lambda_k(Q)$ is the kth Dirichlet eigenvalue of Q. A set Q is said to be quasi–open when there exists a sequence of open sets Q_n for which the capacity of the symmetric difference of Q and Q_n approaches zero. Wolf and Keller [WK] have developed and implemented numerical methods for the minimization of $\Omega \mapsto \lambda_j(\Omega)$ subject to $\lambda_k(\Omega)$ remaining fixed where $\{j,k\} = \{1,2\}$ or $\{2,1\}$.

With respect to the maximization problem, (1.6), we recall that the cruched ice example of Rauch and Taylor [RT] suffices to increase $\lambda_1(\Omega)$ without bound as Ω becomes increasingly perforated. Hence, some geometrical constraints must be imposed if (1.6) is to have sense. Cox and Ross [CR] have demonstrated that $\Omega \mapsto \lambda_k(\Omega)$ attains its extremes over those planar Ω of fixed area whose perimeter is less than a prescibed value and that are starlike with respect to a common disk.

The discussion presented through §5 is not limited to two phase conductivity. These ideas carry over to extremal eigenvalue problems for two phase elastic structures in three dimensions. The relaxation of the elasticity problem proceeds along the same lines as presented for two phase conductivity. One finds that the optimal relaxed design lies within the class of finite rank laminates; this follows from a comparison principle introduced by Avellaneda [Av] together with the min max characterization of eigenvalues. Necessary conditions of the type found in §§3 and 4 may be obtained provided one has intrinsic formulas for the effective tensors of laminar microstructures analogous to those given by (2.1) and (2.2). Such representations follow from the work of Francfort, Murat, and Tartar [FMT]. Indeed, we consider three dimensional mixtures of two well ordered elastic components with elasticities C_1 and C_2 in the proportions θ and $1-\theta$. Restricting to probability measures, application of a slight modification of Theorem 2.1 of [FMT] shows that the set of effective elastic tensors $\overline{C}$ and $\underline{C}$ for laminates is equivalent to the set of tensors given by

$$\overline{C}(\theta,T) = C_2 - \theta[(C_2 - C_1)^{-1} - (1-\theta)L_2(T)]^{-1}$$
$$\underline{C}(\theta,T) = C_1 + (1-\theta)[(C_2 - C_1)^{-1} + \theta L_1(T)]^{-1}.$$

where L_1 and L_2 are linear transformations from totally symmetric fourth order tensors into elasticity tensors and T is a totally symmetric positive semidefinite fourth order tensor with unit trace.

7. **References**

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