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WHERE BEST TO HOLD A DRUM FAST*

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To John Dennis, progenitor, advocate, and friend, on his 60th birthday

Abstract. If we are allowed to fasten, say, one half of a drum's boundary, which half produces the lowest or highest bass note? The answer is a natural limit of solutions to a family of extremal Robin problems for the least eigenvalue of the Laplacian. We produce explicit extremizers when the drum is a disk while for general shapes we establish existence and necessary conditions, and we build and test a pair of numerical methods.

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1. Introduction.

We consider the fundamental mode of vibration of a drumhead that is fastened along part of its boundary and free on the remainder. More precisely, we study the least eigenvalue of

$$\begin{aligned} -\Delta u &= \xi u \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \Gamma, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma, \end{aligned}$$

where Ω is a smooth, open, bounded, connected planar set and Γ is a measurable subset of its boundary. We denote this least eigenvalue by $\xi_1(\Gamma)$ and seek its extremes as Γ varies over subsets of $\partial\Omega$ of prescribed measure. Closely related questions for one-dimensional continua have been raised in the engineering literature; see, e.g., Mroz and Rozvany [14] and Chuang and Hou [5].

We begin the analysis of our model problem by expressing the two boundary conditions in the single equation

$$1_\Gamma u + (1 - 1_\Gamma)\partial u/\partial n = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where 1_Γ denotes the characteristic function of Γ. With an eye toward a convenient variational characterization of $\xi_1(\Gamma)$ we note that (1.1) is not a boundary condition of the third (or Robin) type. To achieve this the coefficient of $\partial u/\partial n$ must be constant. Before blindly dividing through by $1 - 1_\Gamma$ we introduce a simple regularization. In particular, we arrive at (1.1) in the limit as $\varepsilon \to 0$ in

$$1_\Gamma u + (1 + \varepsilon - 1_\Gamma)\partial u/\partial n = 0 \quad \text{on} \quad \partial\Omega$$

or, equivalently,

$$\varepsilon^{-1} 1_\Gamma u + \partial u/\partial n = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

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Physically, the drumhead remains free on $\partial\Omega \setminus \Gamma$ while on Γ it is elastically supported by a fastener of stiffness $1/\varepsilon$. We denote by $\xi_1^\varepsilon(\Gamma)$ the least eigenvalue of $-\Delta$ subject to (1.2). This boundary condition is indeed of the third type and so we may record the weak formulation

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \varepsilon^{-1} \int_{\partial\Omega} 1_\Gamma uv \, ds = \xi \int_\Omega uv \, dx \qquad \forall\, v \in H^1(\Omega) \tag{1.3}$$

and the associated variational characterization

$$\xi_1^\varepsilon(\Gamma) = \inf_{u \in H_1^1(\Omega)} \int_\Omega |\nabla u|^2 \, dx + \varepsilon^{-1} \int_{\partial\Omega} 1_\Gamma u^2 \, ds, \tag{1.4}$$

where $H_1^1(\Omega)$ is the class of $H^1(\Omega)$ functions with $L^2(\Omega)$ norm one. The advantage of the chosen regularization lies in the fact that in both (1.3) and (1.4), the underlying function space *does not* vary with Γ.

We now fix a number $\gamma \in (0,1)$ (the Dirichlet fraction) and formulate the optimal design problems whose solutions will determine the range of $\xi_1^\varepsilon(\Gamma)$ as Γ varies over those subsets of $\partial\Omega$ of size $\gamma|\partial\Omega|$. In particular, we study

$$\inf_{1_\Gamma \in ad_\gamma(\partial\Omega)} \xi_1^\varepsilon(\Gamma) \quad \text{and} \quad \sup_{1_\Gamma \in ad_\gamma(\partial\Omega)} \xi_1^\varepsilon(\Gamma),$$

where

$$ad_\gamma(\partial\Omega) \equiv \{1_\Gamma : \Gamma \subset \partial\Omega,\ |\Gamma| = \gamma|\partial\Omega|\}$$

and $|\Gamma|$ denotes the one-dimensional Hausdorff measure of Γ. Generally speaking, we shall see that minimal designs favor a connected Γ while maximal designs tend to fragment Γ. Accordingly, in section 2, we establish existence of minimizers and (relaxed) maximizers by showing that ξ_1^ε is weak* continuous on the weak* closure of $ad_\gamma(\partial\Omega)$. In section 3 we characterize minimizers via first order necessary conditions and provide an explicit minimal design for the disk. In section 4 analogous first order conditions lead to the uniqueness of the maximizer and its characterization in terms of the normal derivative of the first eigenfunction of the pure Dirichlet problem. In section 5 we construct distinct approaches to the numerical minimization and maximization of ξ_1^ε. We test these methods on elliptical and L-shaped drums in section 6.

Although stated in the context of the planar Laplacian, our arguments apply, without change, to second order self-adjoint elliptic equations on smooth bounded domains in an arbitrary number of dimensions. Although isoperimetric inequalities for mixed and Robin problems have received considerable attention (see, e.g., Bandle [1]) the paper of Buttazzo [4] appears to be the first and only to consider an extremal Robin problem on a fixed domain.

On completion of this work we learned that Denzler [10] had been simultaneously pursuing the same set of questions. Via methods quite distinct from those invoked here he showed that ξ_1 attains its minimum on $ad_\gamma(\partial\Omega)$ and that the supremum of ξ_1 is $\lambda_1(\Omega)$, the least Dirichlet eigenvalue.

2. Existence. We shall denote by $L(\partial\Omega, [0,1])$ those measurable functions on $\partial\Omega$ that take values in the interval $[0,1]$. With respect to the weak* topology on $L^\infty(\partial\Omega)$ Friedland [12] has shown the following.

Proposition 2.1. *The weak* closure of* $ad_\gamma(\partial\Omega)$ *is*

$$ad_\gamma^*(\partial\Omega) \equiv \left\{ \theta \in L(\partial\Omega, [0,1]) : \int_{\partial\Omega} \theta(x) \, ds = \gamma|\partial\Omega| \right\}.$$

In addition, $ad_\gamma(\partial\Omega)$ is the set of extreme points of $ad^*_\gamma(\partial\Omega)$.

For $\theta \in ad^*_\gamma(\partial\Omega)$ we denote by $\xi^\varepsilon_1(\theta)$ the first eigenvalue of $-\Delta$ subject to

$$\varepsilon^{-1}\theta u + \partial u/\partial n = 0 \quad \text{on} \quad \partial\Omega. \tag{2.1}$$

The analogous variational characterization

$$\xi^\varepsilon_1(\theta) = \inf_{u\in H^1_1(\Omega)} \mathcal{R}_\varepsilon(u,\theta), \qquad \text{where} \quad \mathcal{R}_\varepsilon(u,\theta) \equiv \int_\Omega |\nabla u|^2\,dx + \varepsilon^{-1}\int_{\partial\Omega} \theta u^2\,ds, \tag{2.2}$$

leads immediately to

$$0 < \xi^\varepsilon_1(\theta) \le \lambda_1(\Omega) \qquad \forall\,\theta \in ad^*_\gamma(\partial\Omega) \quad \text{and} \quad \forall\,\varepsilon > 0, \tag{2.3}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ subject to Dirichlet conditions over the *entire* boundary. As $\mathcal{R}_\varepsilon(u,\theta) = \mathcal{R}_\varepsilon(|u|,\theta)$ it follows from (2.2) that $\xi^\varepsilon_1(\theta)$ is simple and may be associated with a nonnegative eigenfunction.

PROPOSITION 2.2. *The mapping $\theta \mapsto \xi^\varepsilon_1(\theta)$ is continuous with respect to the weak* topology on $L(\partial\Omega,[0,1])$.*

Proof. Suppose $\theta_n \overset{*}{\rightharpoonup} \theta$ and that u_n is the positive first eigenfunction, associated with θ_n, normalized such that

$$\int_\Omega u_n^2\,dx = 1 \quad \text{and} \quad \int_\Omega |\nabla u_n|^2\,dx + \varepsilon^{-1}\int_{\partial\Omega} \theta_n u_n^2\,ds = \xi^\varepsilon_1(\theta_n). \tag{2.4}$$

From (2.3) and (2.4) it follows that $\{u_n\}_n$ is bounded in $H^1(\Omega)$ and hence that $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$ and the traces $u_n|_{\partial\Omega} \to u|_{\partial\Omega}$ in $L^2(\partial\Omega)$. In addition $\xi^\varepsilon_1(\theta_n) \to \xi$. These observations permit us to pass to the limit in the weak form

$$\int_\Omega \nabla u_n \cdot \nabla v\,dx + \varepsilon^{-1}\int_{\partial\Omega} \theta_n u_n v\,ds = \xi^\varepsilon_1(\theta_n)\int_\Omega u_n v\,dx$$

and so conclude that ξ and u constitute an eigenpair for θ. As u is positive it follows that $\xi = \xi^\varepsilon_1(\theta)$. ☐

As $ad^*_\gamma(\partial\Omega)$ is weak* compact Corollary 2.3 now follows.

COROLLARY 2.3.

$$\inf_{1_\Gamma \in ad_\gamma(\partial\Omega)} \xi^\varepsilon_1(\Gamma) = \min_{\theta\in ad^*_\gamma(\partial\Omega)} \xi^\varepsilon_1(\theta)$$

and

$$\sup_{1_\Gamma \in ad_\gamma(\partial\Omega)} \xi^\varepsilon_1(\Gamma) = \max_{\theta\in ad^*_\gamma(\partial\Omega)} \xi^\varepsilon_1(\theta).$$

Our interest is in characterizing those θ at which ξ^ε_1 attains its extremes. A number of previous studies have produced lower and upper bounds for $\xi^\varepsilon_1(\theta)$.

Regarding the latter, such bounds are typically achieved by replacing θ with a constant and Ω with a disk. Pólya and Szegő accomplish this for starlike Ω via the method of similar level lines; see Bandle [1, Thm. III.3.21]. Hersch uses conformal transplantation and so requires that Ω merely be simply connected. More precisely, he demonstrates (see [1, Thm. III.3.17]) that

$$\xi^\varepsilon_1(\theta,\Omega) \le \xi^\varepsilon_1(\gamma|\partial\Omega|/|\partial D_\Omega|, D_\Omega) \qquad \forall\,\theta \in ad^*_\gamma(\partial\Omega), \tag{2.5}$$

where D_Ω is the disk with radius equal to the conformal radius of Ω. Of course, when Ω is itself a disk this result states that $\theta \equiv \gamma$ is maximal.

The construction of useful lower bounds is considerably more difficult. All attempts to bound $\xi_1^\varepsilon(\theta)$ from below apply *only* to the case of constant θ. We cite Philippin [15], Bossel [3], and Sperb [17].

3. Minimizing ξ_1^ε. We show that $\theta \mapsto \xi_1^\varepsilon(\theta)$ possesses a classical, i.e., $ad_\gamma(\partial\Omega)$, minimizer. We compute it in the case of the disk while in the general case we produce pointwise optimality conditions.

Returning to (2.2) we recognize that $\theta \mapsto \xi_1^\varepsilon(\theta)$ is an infimum of affine functions of θ. The following proposition results.

PROPOSITION 3.1. *$\theta \mapsto \xi_1^\varepsilon(\theta)$ is concave on $ad_\gamma^*(\partial\Omega)$.*

If we now recall (see, e.g., Bauer [2]) that a bounded concave function on a compact convex set attains its minimum at an extreme point, we arrive at the following.

COROLLARY 3.2. *$\theta \mapsto \xi_1^\varepsilon(\theta)$ attains its minimum on $ad_\gamma(\partial\Omega)$.*

We now produce an explicit minimizer in the case that Ω is a disk, D. This is accomplished through circular symmetrization, defined as follows.

Given $v \in H^1(D)$ we take $u(r,t) = v(x)$, where $x = r(\cos t, \sin t)$ and $-\pi < t \le \pi$. Now, at each r we replace $t \mapsto u(r,t)$ with its symmetrically increasing rearrangement

$$u^\vee(r,t) = \inf\{c : t \in \{s : u(r,s) \le c\}^*\},$$

where A^* is simply the interval $(-|A|/2, |A|/2)$. We then take $v^\vee(x) \equiv u^\vee(r,t)$ to be the circular (increasing about $t = 0$) rearrangement of v. The corresponding symmetrically decreasing rearrangement is

$$u^\wedge(r,t) = u^\vee(r, \pi - |t|).$$

As a simple example we note that if $1_\Gamma \in ad_\gamma(\partial D)$, then

$$1_\Gamma^\wedge(t) = 1_{\Gamma^*} = \begin{cases} 1 & \text{if } |t| \le \gamma\pi, \\ 0 & \text{otherwise.} \end{cases}$$

We now recall (see, e.g., Cox and Kawohl [9]) that circular rearrangement cannot increase the Dirichlet integral and that $u^\vee$ and $1_\Gamma^\wedge$ are oppositely ordered. As a result,

$$\mathcal{R}_\varepsilon(v, 1_\Gamma) \ge \mathcal{R}_\varepsilon(v^\vee, 1_{\Gamma^*}) \qquad \forall\,(v, 1_\Gamma) \in H_1^1(D) \times ad_\gamma(\partial D)$$

and so we arrive at the following proposition.

PROPOSITION 3.3. *$1_\Gamma \mapsto \xi_1^\varepsilon(1_\Gamma)$ attains its minimum at 1_{Γ^*}.*

As 1_{Γ^*} is clearly independent of ε we proceed to let ε approach 0. Our preliminary result does not require the domain to be a disk.

LEMMA 3.4. *If $\Gamma \subset \partial\Omega$, then $\xi_1^\varepsilon(1_\Gamma) \to \xi_1(1_\Gamma)$ as $\varepsilon \to 0$.*

Proof. Let $u_\varepsilon \in H_1^1(\Omega)$ denote the eigenfunction associated with $\xi_1^\varepsilon(1_\Gamma)$. Now, recalling (2.3), we find

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx + \varepsilon^{-1}\int_\Gamma u_\varepsilon^2\,dx = \xi_1^\varepsilon(1_\Gamma) \le \lambda_1(\Omega). \tag{3.1}$$

As a result, $\{u_\varepsilon\}_{\varepsilon>0}$ is clearly bounded in $H^1(\Omega)$ and, moreover,

$$\int_\Gamma u_\varepsilon^2\,dx = O(\varepsilon).$$

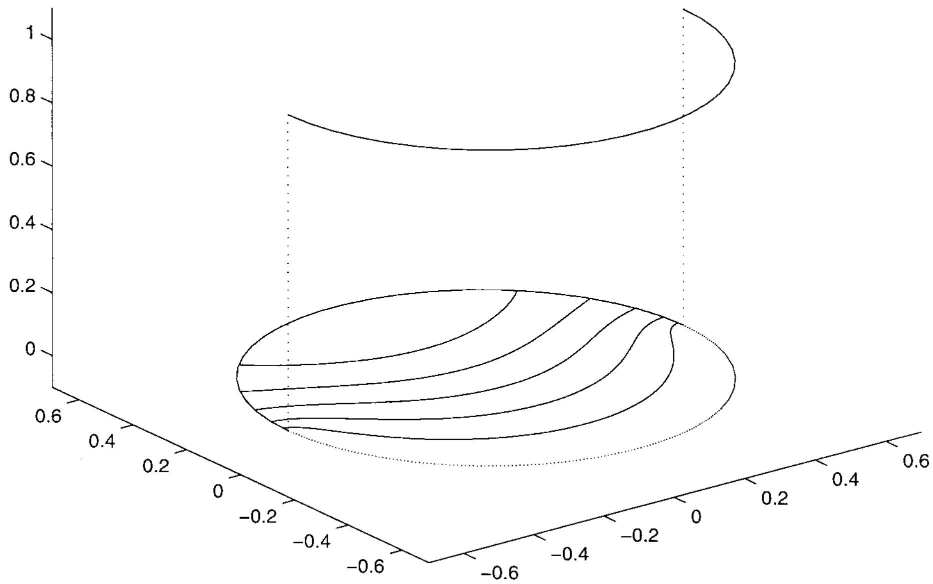

FIG. 1. *Minimal fastening of the disk.*

Hence (a subsequence of) u_ε converges weakly in $H^1(\Omega)$ to some $u_0 \in H_1^1(\Omega, \Gamma)$, those functions in $H_1^1(\Omega)$ with vanishing trace on Γ. We now show that u_0 is the eigenfunction associated with $\xi_1(1_\Gamma)$. Taking the limit inferior throughout (3.1) gives

$$\int_\Omega |\nabla u_0|^2 \, dx \le \liminf_{\varepsilon \to 0} \xi_1^\varepsilon(\Gamma).$$

Now if there exists a $u \in H_1^1(\Omega, \Gamma)$ and a $\delta > 0$ for which

$$\int_\Omega |\nabla u|^2 \, dx \le \int_\Omega |\nabla u_0|^2 \, dx - \delta,$$

then (3.1) implies $\mathcal{R}_\varepsilon(u, 1_\Gamma) < \xi_1^\varepsilon(\Gamma)$ for some ε, contrary to Rayleigh's principle. Hence,

$$\xi_1(\Gamma) = \int_\Omega |\nabla u_0|^2 \, dx \le \liminf_{\varepsilon \to 0} \xi_1^\varepsilon(\Gamma).$$

The simple observation $\xi_1^\varepsilon(\Gamma) \le \xi_1(\Gamma)$ completes the argument. □

COROLLARY 3.5. $1_\Gamma \mapsto \xi_1(1_\Gamma)$ *attains its minimum at* 1_{Γ^*}.

In Figure 1 we have plotted 1_{Γ^*} for $\gamma = 1/2$ on the disk of unit diameter along with the contours of the associated first eigenfunction, computed by the `pdeeig` routine in MATLAB [13] via a piecewise linear approximation on 259328 triangles. The computed value of $\xi_1(1_{\Gamma^*})$ is 4.86.

As the eigenvalue problem for such a design does not yield to separation of variables we return to the question posed at the close of the last section, namely, can one bound $\xi_1(1_{\Gamma^*})$ from below? Even in this simplest of all possible geometries our best analytical bound requires the majority of the boundary to be Dirichlet. More precisely, if Ω is the disk of radius R and $\gamma > 1/2$, then

$$\xi_1(1_{\Gamma^*}) \ge \frac{2\gamma - 1}{2R^2} j_0^2,$$

where j_0 is the first zero of the Bessel function J_0. This follows from Bandle's generalization of a result of Nehari; see [1, Thm. III.3.9].

We now return to a general domain and denote by $\check\theta^\varepsilon$ the minimizer of ξ_1^ε over $ad_\gamma^*(\partial\Omega)$. We take $\check u^\varepsilon \in H_1^1(\Omega)$ to be the positive eigenfunction associated with $\check\theta^\varepsilon$ and record

$$\xi_1^\varepsilon(\check\theta^\varepsilon) = \mathcal{R}_\varepsilon(\check u^\varepsilon, \check\theta^\varepsilon) = \min_{\theta\in ad_\gamma^*(\partial\Omega)} \min_{u\in H_1^1(\Omega)} \mathcal{R}_\varepsilon(u,\theta) = \min_{u\in H_1^1(\Omega)} \min_{\theta\in ad_\gamma^*(\partial\Omega)} \mathcal{R}_\varepsilon(u,\theta).$$

In other words,

$$\mathcal{R}_\varepsilon(\check u^\varepsilon, \check\theta^\varepsilon) = \min_{u\in H_1^1(\Omega)} \mathcal{R}_\varepsilon(u, \check\theta^\varepsilon) \quad \text{and} \quad \mathcal{R}_\varepsilon(\check u^\varepsilon, \check\theta^\varepsilon) = \min_{\theta\in ad_\gamma^*(\partial\Omega)} \mathcal{R}_\varepsilon(\check u^\varepsilon, \theta).$$

The former simply states that $\check u^\varepsilon$ is an eigenfunction corresponding to $\check\theta^\varepsilon$. The latter, however, informs us that

$$\int_{\partial\Omega} \check\theta^\varepsilon |\check u^\varepsilon|^2\, ds = \min_{\theta\in ad_\gamma^*(\partial\Omega)} \int_{\partial\Omega} \theta |\check u^\varepsilon|^2\, ds. \tag{3.2}$$

We remove the integral constraint on $\check\theta^\varepsilon$ at the cost of a Lagrange multiplier. More precisely, from the Lagrange multiplier rule, [6, Thm. 6.1.1], we deduce that (3.2) implies the existence of $\nu_1 \ge 0$ and $|\nu_1| + |\nu_2| > 0$ such that

$$\int_{\partial\Omega} \check\theta^\varepsilon (\nu_1 |\check u^\varepsilon|^2 + \nu_2)\, ds = \min_{\theta\in L(\partial\Omega,[0,1])} \int_{\partial\Omega} \theta(\nu_1 |\check u^\varepsilon|^2 + \nu_2)\, ds. \tag{3.3}$$

From $\nu_1|\check u^\varepsilon|^2 \ge 0$ we deduce from (3.3) that $\nu_2 \le 0$.

If $\nu_2 = 0$, then (3.3) implies that $\check\theta^\varepsilon \check u^\varepsilon$ must vanish on the full boundary. Now, the boundary condition (2.1) implies that $\check u^\varepsilon$ is a Neumann eigenfunction. As $\check u^\varepsilon$ does not change sign it can only be the constant eigenfunction. Now $\check\theta^\varepsilon \check u^\varepsilon = 0$ implies that $\check\theta^\varepsilon$ is identically zero, contrary to its integral constraint. Therefore, $\nu_2 < 0$.

Now, if $\nu_1 = 0$, then as $\nu_2 < 0$, (3.3) implies that $\check\theta^\varepsilon$ is identically one, contrary to its integral constraint. Therefore, $\nu_1 > 0$.

With $\nu^2 \equiv -\nu_2/\nu_1$ we deduce from (3.3) the following pointwise necessary conditions:

$$\check\theta^\varepsilon(x) = 0 \Rightarrow \check u^\varepsilon(x) \ge \nu, \tag{3.4}$$

$$0 < \check\theta^\varepsilon(x) < 1 \Rightarrow \check u^\varepsilon(x) = \nu, \tag{3.5}$$

$$\check\theta^\varepsilon(x) = 1 \Rightarrow \check u^\varepsilon(x) \le \nu. \tag{3.6}$$

Recalling that $\check\theta^\varepsilon$ may be assumed a member of $ad_\gamma(\partial\Omega)$, it follows that $\check\theta^\varepsilon$ jumps across a level set of the trace of its corresponding eigenfunction, $\check u^\varepsilon$.

4. Maximizing ξ_1^ε.

Recalling (2.5) we begin with a simple proof of the fact that constant θ is maximal for the disk. Noting only that u_γ, the eigenfunction corresponding to $\theta \equiv \gamma$ on the disk, is radial we find

$$\xi_1^\varepsilon(\theta) \le \mathcal{R}_\varepsilon(u_\gamma, \theta) = \mathcal{R}_\varepsilon(u_\gamma, \gamma) = \xi_1^\varepsilon(\gamma) \qquad \forall\, \theta \in ad_\gamma^*(\partial D). \tag{4.1}$$

With regard to general Ω we shall see that where the maximizing θ is neither zero nor one the trace of its corresponding eigenfunction is, like u_γ, constant. In addition, we establish uniqueness of the maximizer and show that when it lies everywhere between zero and one it is (to lowest order in ε) proportional to the normal derivative of the first Dirichlet eigenfunction on Ω.

The first step is the derivation of pointwise conditions analogous to (3.4)–(3.6). These shall stem from knowledge of the gradient of $\theta \mapsto \xi_1^\varepsilon(\theta)$.

PROPOSITION 4.1. *$\theta \mapsto \xi_1^\varepsilon(\theta)$ is smooth and*

$$\langle \partial\xi_1^\varepsilon(\theta), \psi\rangle = \varepsilon^{-1}\int_{\partial\Omega}\psi u^2\,ds,$$

where $u \in H_1^1(\Omega)$ is the nonnegative eigenfunction associated with θ.

Proof. The gradient of a simple eigenvalue of a self-adjoint operator is the gradient of the Rayleigh quotient evaluated at the corresponding eigenfunction. See Cox [8] for details. ∎

If $\hat\theta^\varepsilon$ maximizes ξ_1^ε over $ad_\gamma^*(\partial\Omega)$, then $\partial\xi_1^\varepsilon(\hat\theta^\varepsilon) \in N_{ad_\gamma^*(\partial\Omega)}(\hat\theta^\varepsilon)$, the cone of normals to $ad_\gamma^*(\partial\Omega)$ at $\hat\theta^\varepsilon$. As $ad_\gamma^*(\partial\Omega)$ is convex this means that

$$\langle \partial\xi_1^\varepsilon(\hat\theta^\varepsilon), \hat\theta^\varepsilon\rangle = \max_{\theta\in ad_\gamma^*(\partial\Omega)}\langle \partial\xi_1^\varepsilon(\hat\theta^\varepsilon), \theta\rangle,$$

that is,

$$\int_{\partial\Omega}\hat\theta^\varepsilon|\hat u^\varepsilon|^2\,ds = \max_{\theta\in ad_\gamma^*(\partial\Omega)}\int_{\partial\Omega}\theta|\hat u^\varepsilon|^2\,ds, \tag{4.2}$$

where $\hat u^\varepsilon$ is the positive eigenfunction corresponding to $\hat\theta^\varepsilon$. As above, to shed the integral constraint we invoke the Lagrange multiplier rule of Clarke. This gives a $\nu_1 \le 0$ and ν_2 for which $|\nu_1| + |\nu_2| > 0$ and

$$\int_{\partial\Omega}\hat\theta^\varepsilon(\nu_1|\hat u^\varepsilon|^2 + \nu_2)\,ds = \max_{\theta\in L(\partial\Omega,[0,1])}\int_{\partial\Omega}\theta(\nu_1|\hat u^\varepsilon|^2+\nu_2)\,ds. \tag{4.3}$$

From $\nu_1|\hat u^\varepsilon|^2 \le 0$ we deduce from (4.3) that $\nu_2 > 0$. Similarly, $\nu_1 < 0$. With $\nu^2 \equiv -\nu_2/\nu_1$ we arrive at the pointwise necessary conditions

$$\hat\theta^\varepsilon(x) = 0 \Rightarrow \hat u^\varepsilon(x) \le \nu, \tag{4.4}$$

$$0 < \hat\theta^\varepsilon(x) < 1 \Rightarrow \hat u^\varepsilon(x) = \nu, \tag{4.5}$$

$$\hat\theta^\varepsilon(x) = 1 \Rightarrow \hat u^\varepsilon(x) \ge \nu. \tag{4.6}$$

From Proposition 3.1 we note that these conditions are also sufficient.

A further consequence of (4.2) is that $(\hat u^\varepsilon, \hat\theta^\varepsilon)$ is a saddle point of $\mathcal{R}_\varepsilon$, i.e.,

$$\mathcal{R}_\varepsilon(\hat u^\varepsilon,\theta) \le \mathcal{R}_\varepsilon(\hat u^\varepsilon,\hat\theta^\varepsilon) \le \mathcal{R}_\varepsilon(u,\hat\theta^\varepsilon) \qquad \forall\,(u,\theta)\in H_1^1(\Omega)\times ad_\gamma^*(\partial\Omega).$$

From this observation comes the following proposition.

PROPOSITION 4.2. *$\hat\theta^\varepsilon$ is unique.*

Proof. Suppose that θ_1 and θ_2 are both maximizers of $\theta \mapsto \xi_1^\varepsilon(\theta)$ and that u_1 and u_2 are the respective first eigenfunctions. We find

$$\mathcal{R}_\varepsilon(u_1,\theta_2) \le \mathcal{R}_\varepsilon(u_1,\theta_1) \le \mathcal{R}_\varepsilon(u_2,\theta_1),$$
$$\mathcal{R}_\varepsilon(u_2,\theta_1) \le \mathcal{R}_\varepsilon(u_2,\theta_2) \le \mathcal{R}_\varepsilon(u_1,\theta_2).$$

However, as $\mathcal{R}_\varepsilon(u_1,\theta_1)=\mathcal{R}_\varepsilon(u_2,\theta_2)$ we find that u_1 and u_2 are both eigenfunctions for θ_1 and hence $u_1=u_2$. Recalling the respective weak forms we find

$$\int_{\partial\Omega}(\theta_1-\theta_2)u_1 v\,ds=0 \qquad \forall\, v\in H^1(\Omega),$$

and hence $\theta_1=\theta_2$ on the support of $u_1|_{\partial\Omega}$, the trace of u_1. Off of the support of $u_1|_{\partial\Omega}$ it follows from (4.4) that $\theta_1=\theta_2=0$. □

From uniqueness we are able to ascertain symmetry. In particular, if Ω is symmetric with respect to a line ℓ we may reflect $\hat\theta^\varepsilon$ across ℓ to $\hat\theta^\varepsilon_\ell$. By simply reflecting the associated $\hat u^\varepsilon$ it follows that $\xi^\varepsilon_1(\hat\theta^\varepsilon)=\xi^\varepsilon_1(\hat\theta^\varepsilon_\ell)$ and hence, by uniqueness, that $\hat\theta^\varepsilon=\hat\theta^\varepsilon_\ell$. We have proven the following.

PROPOSITION 4.3. *$\hat\theta^\varepsilon$ is symmetric about every line of symmetry of Ω.*

This leads to a third proof of (4.1).

PROPOSITION 4.4. *If Ω is a disk, then $\hat\theta^\varepsilon\equiv\gamma$. Disks are the only (smooth) sets with a constant maximizer.*

Proof. Full symmetry implies that $\hat\theta^\varepsilon$ must be constant. The only admissible constant is γ. Given a constant maximizer, it follows from (4.5) that $\check u^\varepsilon$ is identically ν on $\partial\Omega$. From the boundary condition (2.1) we then find that $\partial\check u^\varepsilon/\partial n=-\nu\gamma/\varepsilon$ on $\partial\Omega$. Serrin [16, Thm. 2] has shown that a disk is the *only* C^2 domain on which one may solve $(\Delta+\xi)u=0$ subject to constant Dirichlet and Neumann data. □

If $\Omega=D_a$ is a disk of radius a, then $u(r)=J_0(\sqrt{\xi}r)$ is a radial solution of $-\Delta u=\xi u$. The best eigenvalue, $\xi^\varepsilon_1(\gamma)$, is therefore the least positive ξ for which

$$\gamma u(a)+\varepsilon u'(a)=0.$$

It follows immediately then that $\xi^\varepsilon_1(\gamma)\to\lambda_1(D_a)$ as $\varepsilon\to 0$, where $\lambda_1(D_a)$ is the least positive root of $\lambda\mapsto J_0(\sqrt{\lambda}a)$, i.e., the first Dirichlet eigenvalue of D_a. This approach to the Dirichlet eigenvalue holds, in fact, for every domain Ω.

PROPOSITION 4.5. *If $\hat\theta^\varepsilon$ maximizes $\theta\mapsto\xi^\varepsilon_1(\theta)$ over $ad_\gamma(\partial\Omega)$, then $\xi^\varepsilon_1(\hat\theta^\varepsilon)\to\lambda_1(\Omega)$ as $\varepsilon\to 0$.*

Proof. As $\xi^\varepsilon_1(\gamma)\le\xi^\varepsilon_1(\hat\theta^\varepsilon)\le\lambda_1(\Omega)$ it suffices to show that

$$\lambda_1(\Omega)\le\liminf_{\varepsilon\to 0}\xi^\varepsilon_1(\gamma). \tag{4.7}$$

Let us denote by $u^\varepsilon_1\in H^1_1(\Omega)$ the positive eigenfunction corresponding to $\xi^\varepsilon_1(\gamma)$. As $\|u^\varepsilon_1\|_2=1$ and $\|\nabla u^\varepsilon_1\|_2^2\le\lambda_1(\Omega)$ it follows that there exists a $u_1\in H^1_1(\Omega)$ for which $u^\varepsilon_1\rightharpoonup u_1$ in $H^1(\Omega)$ as $\varepsilon\to 0$. Given the normalization of u^ε_1 we find that

$$\gamma\int_{\partial\Omega}|u^\varepsilon_1|^2\,ds=\varepsilon\int_\Omega|\nabla u^\varepsilon_1|^2\,dx+\varepsilon\xi^\varepsilon_1(\gamma)\to 0$$

as $\varepsilon\to 0$, i.e., $u^\varepsilon_1|_{\partial\Omega}\to 0$ in $L^2(\partial\Omega)$. As $u^\varepsilon_1|_{\partial\Omega}\to u_1|_{\partial\Omega}$ in $L^2(\partial\Omega)$ it follows that $u_1\in H^1_0(\Omega)$. Now, given the weak lower semicontinuity of $u\mapsto\|\nabla u\|_2^2$ and the nonnegativity of the boundary term, we find

$$\int_\Omega|\nabla u_1|^2\,dx\le\liminf_{\varepsilon\to 0}\int_\Omega|\nabla u^\varepsilon_1|^2\,dx+\frac{\gamma}{\varepsilon}\int_{\partial\Omega}|u^\varepsilon_1|^2\,ds=\liminf_{\varepsilon\to 0}\xi^\varepsilon_1(\gamma).$$

As $u_1\in H^1_0(\Omega)$ and $\|u^\varepsilon_1\|_2=1$ it follows from Rayleigh's principle that the left-hand side is larger than $\lambda_1(\Omega)$. This establishes (4.7). □

This proposition addresses the limiting behavior of the eigenvalue but says nothing about the limiting optimal design. We shall now show that if the limiting design takes values strictly between 0 and 1, then it is proportional to the normal derivative of the first Dirichlet eigenfunction.

We begin at the necessary condition (4.5) and note that for constant ν and $\xi < \lambda_1(\Omega)$ one may solve

$$-\Delta u = \xi u \quad \text{in} \quad \Omega, \qquad u = \nu \quad \text{on} \quad \partial\Omega$$

in terms of the Dirichlet eigenfunctions, $\{\phi_j\}$, and Dirichlet eigenvalues, $\{\lambda_j\}$, of Ω. In particular,

$$u = \nu + \nu\xi \sum_{j=1}^{\infty} \frac{\langle \phi_j, 1\rangle}{\lambda_j - \xi}\phi_j.$$

The Robin condition (2.1) now suggests

$$\theta = -\frac{\varepsilon}{\nu}\frac{\partial u}{\partial n} = -\varepsilon\xi \sum_{j=1}^{\infty} \frac{\langle \phi_j, 1\rangle}{\lambda_j - \xi}\frac{\partial \phi_j}{\partial n}. \tag{4.8}$$

Integrating this expression over $\partial\Omega$ we find

$$\gamma|\partial\Omega| = \int_{\partial\Omega} \theta\, ds = -\varepsilon\xi \sum_{j=1}^{\infty} \frac{\langle \phi_j, 1\rangle}{\lambda_j - \xi}\int_{\partial\Omega} \frac{\partial \phi_j}{\partial n}\, ds = \varepsilon\xi \sum_{j=1}^{\infty} \frac{\langle \phi_j, 1\rangle^2}{\lambda_j - \xi}\lambda_j. \tag{4.9}$$

We view this as an equation for ξ. As the right side is continuous and strictly increasing from 0 (at $\xi = 0$) to ∞ (at $\xi = \lambda_1(\Omega)$) there exists a unique solution, ξ_1^ε, depending smoothly on ε. Expressing ξ_1^ε as a power series, identification of like powers in (4.9) brings

$$\xi_1^\varepsilon = \lambda_1(\Omega) - \frac{\lambda_1^2(\Omega)\langle \phi_1, 1\rangle^2}{\gamma|\partial\Omega|}\varepsilon + O(\varepsilon^2). \tag{4.10}$$

Substituting this into (4.8) we arrive at

$$\theta^\varepsilon = \gamma \frac{\partial \phi_1}{\partial n}\Big/\overline{\frac{\partial \phi_1}{\partial n}} + O(\varepsilon) \quad \text{where} \quad \overline{\frac{\partial \phi_1}{\partial n}} = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \frac{\partial \phi_1}{\partial n}\, ds. \tag{4.11}$$

Hence, if $\hat{\theta}^\varepsilon$ takes values strictly between 0 and 1 it must necessarily be of this form. Moreover, as the necessary conditions are also sufficient, whenever the above derivation produces an admissible design this design is maximal. Regarding the admissibility of θ^ε we note that, by construction, it is nonnegative and has the correct average. It remains only to check whether it is bounded above by 1. One scenario in which this bound is ensured is when Ω is smooth (in which case $\phi_1 \in C^1(\overline{\Omega})$) and ε and γ are sufficiently small. Finally, we remark that (4.10) provides a nice refinement of Proposition 4.5 in that it expresses, in terms of the Dirichlet fraction, γ, the rate at which $\xi_1^\varepsilon(\hat{\theta}^\varepsilon)$ approaches $\lambda_1(\Omega)$.

5. Algorithms. We confine the design, θ, and the eigenfunction, u, to finite-dimensional spaces and so arrive at optimization problems amenable to a computer.

We write $\partial\Omega$ as the closure of the disjoint union of m open edges, $\{\Gamma_j\}_{j=1}^m$, and then restrict θ to

$$\theta(s) = \sum_{j=1}^m \Theta_j 1_{\Gamma_j}(s),$$

where $\Theta \in \mathbb{R}^m$ satisfies the box constraints

$$0 \le \Theta_j \le 1, \quad j = 1, \dots, m, \tag{5.1}$$

and the integral constraint

$$\sum_{j=1}^m \Theta_j |\Gamma_j| = \gamma |\partial\Omega|. \tag{5.2}$$

To compute ξ_1^ε at such a θ we restrict our search to eigenvectors of the form

$$u(x) = \sum_{i=1}^p U_i T_i(x),$$

where $p < \infty$ and the T_i comprise a so-called Galerkin basis for a p-dimensional subspace of $H^1(\Omega)$. On substituting this expansion into the weak form (1.3) with v running through the T_i we arrive at the $p \times p$ eigensystem

$$(K + \varepsilon^{-1} Q(\Theta))U = \Xi M U, \tag{5.3}$$

where K and M are independent of Θ while

$$Q_{ij}(\Theta) = \int_{\partial\Omega} \theta T_i T_j \, ds = \sum_{k=1}^m \Theta_k \int_{\Gamma_k} T_i T_j \, ds. \tag{5.4}$$

Let us denote the least eigenvalue of (5.3) by $\Xi_1^\varepsilon(\Theta)$. As this approximation procedure respects the symmetry of the original problem we retain a variational characterization,

$$\Xi_1^\varepsilon(\Theta) = \min_{\langle MU,U\rangle=1} \mathcal{R}_\varepsilon(U,\Theta), \qquad \mathcal{R}_\varepsilon(U,\Theta) \equiv \langle (K + \varepsilon^{-1} Q(\Theta))U, U\rangle. \tag{5.5}$$

As $\Theta \mapsto Q(\Theta)$ is linear it follows from (5.5) that $\Theta \mapsto \Xi_1^\varepsilon(\Theta)$ is concave. Now, denoting by AD_γ^* those $\Theta \in \mathbb{R}^m$ satisfying (5.1) and (5.2), we may pose the finite-dimensional optimization problems

$$\min_{\Theta \in AD_\gamma^*} \Xi_1^\varepsilon(\Theta) \quad \text{and} \quad \max_{\Theta \in AD_\gamma^*} \Xi_1^\varepsilon(\Theta).$$

As AD_γ^* is compact and convex and Ξ_1^ε is bounded and concave it follows that $\Theta \mapsto \Xi_1^\varepsilon(\Theta)$ attains its minimum at an extreme point of AD_γ^*, i.e., on AD_γ, those $\Theta \in AD_\gamma^*$ each component of which is either zero or one.

Let us now turn to the gradient of $\Theta \mapsto \Xi_1^\varepsilon(\Theta)$. For well-chosen basis functions, e.g., piecewise linear hats, it can be shown that $\Xi_1^\varepsilon(\Theta) \to \xi_1^\varepsilon(\theta)$ as m and p approach ∞. In particular, $\Xi_1^\varepsilon(\Theta)$ is simple for sufficiently large m and p. As a result we may apply the finite-dimensional analogue of Proposition 4.1,

$$\frac{\partial \Xi_1^\varepsilon(\Theta)}{\partial \Theta_k} = \frac{1}{\varepsilon} \left\langle \frac{\partial Q(\Theta)}{\partial \Theta_k} U_1^\varepsilon, U_1^\varepsilon \right\rangle, \tag{5.6}$$

where the associated eigenvector, U_1^ε, is normalized according to $\langle MU_1^\varepsilon, U_1^\varepsilon\rangle = 1$. The implementation of (5.6), in particular the application of $\partial Q(\Theta)/\partial\Theta_k$, requires a careful accounting of the assembly of Q. Recalling (5.4) we find

$$\frac{\partial Q_{ij}(\Theta)}{\partial\Theta_k} = \int_{\Gamma_k} T_i T_j \, ds.$$

To begin, let us evaluate these integrals under the assumption that Γ_k is the interval $[a,b]$ and that this interval is partitioned by the first components of the grid points $x_i = (s_i, 0)$, i.e.,

$$a = s_1 < s_2 < \cdots < s_{n-1} < s_n = b.$$

We also suppose that $T_i(x_j) = \delta_{ij}$ and that T_i is piecewise linear. As a result

$$\int_{\Gamma_k} T_i T_j \, ds = \frac{1}{3}\begin{cases} |s_1 - s_2| & \text{if } i = j = 1, \\ |s_{i-1} - s_i| + |s_i - s_{i+1}| & \text{if } 1 < i = j < n, \\ |s_{n-1} - s_n| & \text{if } i = j = n, \\ |s_i - s_j|/2 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting the above into (5.6) we find

$$\frac{\partial \Xi_1^\varepsilon(\Theta)}{\partial\Theta_k} = \frac{1}{3\varepsilon}\sum_{i=1}^{n-1}\left\{(U_1^\varepsilon)_i^2 + (U_1^\varepsilon)_i(U_1^\varepsilon)_{i+1} + (U_1^\varepsilon)_{i+1}^2\right\}|s_{i+1} - s_i|.$$

In the general case, i.e., where the T_i remain piecewise linear although Γ_k may be a planar segment whose edges and grid points are ordered by a black-box grid generator (as in MATLAB's PDE toolbox), the gradient takes the form

$$\text{(5.7)} \quad \frac{\partial \Xi_1^\varepsilon(\Theta)}{\partial\Theta_k} = \frac{1}{3\varepsilon}\sum_{i\in\mathcal{I}_k}\langle U_1^\varepsilon\rangle_i|\omega_i|, \qquad \langle U_1^\varepsilon\rangle_i \equiv (U_1^\varepsilon)_{\omega_i^+}^2 + (U_1^\varepsilon)_{\omega_i^+}(U_1^\varepsilon)_{\omega_i^-} + (U_1^\varepsilon)_{\omega_i^-}^2,$$

where $\mathcal{I}_k$ is the set of indices of mesh edges ω_i contained in Γ_k and $\omega_i^\pm$ are the indices of the grid points constituting the endpoints of ω_i. From here it is a simple matter to derive the finite-dimensional analogues of our pointwise optimality conditions. In particular, if each Γ_k corresponds to a single mesh edge and $\check{\Theta}^\varepsilon \in AD_\gamma$ is a classical minimizer of Ξ_1^ε and $\check{U}_1^\varepsilon$ its associated eigenvector, then there exists a ν such that

$$\text{(5.8)} \quad \begin{aligned} \check{\Theta}_k^\varepsilon = 0 &\Rightarrow \langle \check{U}_1^\varepsilon\rangle_k > \nu, \\ \check{\Theta}_k^\varepsilon = 1 &\Rightarrow \langle \check{U}_1^\varepsilon\rangle_k < \nu. \end{aligned}$$

These conditions are reminiscent of those that arise in Krein's problem of the optimal distribution of mass; see, e.g., Cox [7]. As such we apply the simple alternating search strategy of [7] to our minimum problem. More precisely, given $\Theta^{(j)} \in AD_\gamma$,

(I) compute $U^{(j)}$, the minimizer of $U \mapsto \mathcal{R}_\varepsilon(U, \Theta^{(j)})$ subject to $\langle MU, U\rangle = 1$.

(II) compute $\Theta^{(j+1)}$, the minimizer of $\Theta \mapsto \mathcal{R}_\varepsilon(U^{(j)}, \Theta)$ subject to $\Theta \in AD_\gamma$.

(III) if $\Theta^{(j+1)} \neq \Theta^{(j)}$, then set $j = j+1$ and go to (I).

The implementation of (I) simply requires the solution of (5.3) with $\Theta = \Theta^{(j)}$. The optimality conditions (5.8) animate the implementation of (II). More precisely, we compute $\mathcal{J} \equiv \{k : \langle U^{(j)}\rangle_k < \nu\}$, where ν is chosen in such a way that

$$\sum_{k\in\mathcal{J}} |\Gamma_k| = \gamma|\partial\Omega|,$$

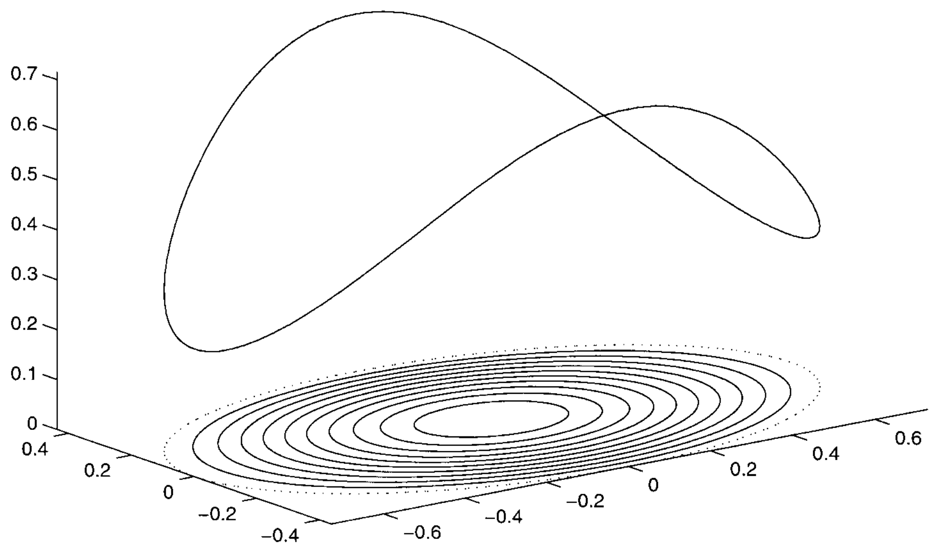

FIG. 2. *The limiting maximal fastener,* Φ.

and then define

$$\Theta_k^{(j+1)} = \begin{cases} 1 & \text{if } k \in \mathcal{J}, \\ 0 & \text{otherwise.} \end{cases}$$

This completes our description of the minimization algorithm.

With respect to the maximization problem, recalling that we have a smooth, concave function subject only to box and linear constraints, we may invoke any of a number of standard optimization packages.

6. Numerical results. For the maximization of Ξ_1^ε we used the `constr` function found in MATLAB's optimization toolbox. The assembly of (5.3) and the computation of Ξ_1^ε and U_1^ε were carried out by the `pdeeig` function found in MATLAB's PDE toolbox. Given U_1^ε we coded the gradient computation (5.7) ourselves. We present here the results of our computations for two representative domains.

In the first case we consider the drumhead whose boundary is the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = \frac{1}{16}.$$

Recalling the discussion at the close of section 4 we expect the maximizer, $\hat{\Theta}^\varepsilon$, as $\varepsilon \to 0$, to coincide with

$$\Phi \equiv \gamma \frac{\partial \phi_1}{\partial n} \Big/ \overline{\frac{\partial \phi_1}{\partial n}},$$

the product of γ and the normalized normal derivative of the first Dirichlet eigenfunction of the ellipse. For the purpose of illustration, in Figure 2 we have plotted the underlying ellipse, the contours of the associated ϕ_1, and the graph of its corresponding Φ, with $\gamma = 1/2$. The eigenfunction was computed at the $p = 96545$ vertices of 191488 triangles. The boundary was partitioned into $m = 100$ edges and the associated Dirichlet eigenvalue was 20.45. Next, we set $\varepsilon = 10^r$, let r range from

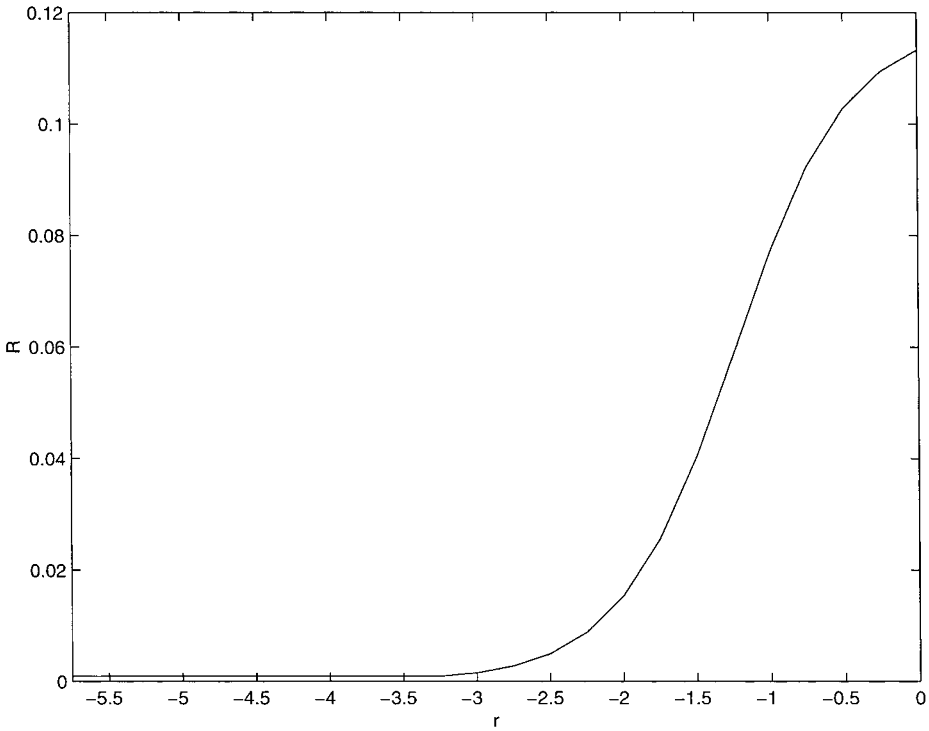

FIG. 3. $\|\Phi - \hat{\Theta}^{\varepsilon}\|_{\infty}$ *as* $\varepsilon \to 0$.

0 to -6, and denote by $\hat{\Theta}^{10^r}$ the maximizer returned by `constr` on the grid quoted above using the default stopping criteria. We measured the pointwise distance from $\hat{\Theta}^{10^r}$ to Φ via

$$R(r) \equiv \|\Phi - \hat{\Theta}^{10^r}\|_{\infty} \equiv \max_k |\Phi_k - \hat{\Theta}_k^{10^r}|$$

and have recorded its graph in Figure 3. That no improvement is seen for $\varepsilon < 10^{-3}$ is most likely due to the fact that our computed Φ is itself accurate only to 10^{-2}.

As a nonconvex example, we pursue the maximizer over the L-shaped region familiar to users of MATLAB. It is well known (see, e.g., Fox, Henrici, and Moler [11]) that the gradient of the first Dirichlet eigenfunction is not bounded in a neighborhood of the reentrant corner. As a result, we may not expect (4.11) to hold along the entire boundary. In Figure 4 we have plotted $\hat{\Theta}^{\varepsilon}$, the maximizer returned by `constr` along with the level sets of its corresponding eigenfunction. Working over a grid of $p = 49665$ vertices, 97792 triangles, and $m = 192$ boundary segments with $\varepsilon = 10^{-3}$ and $\gamma = 1/2$ we found $\xi_1^{\varepsilon}(\hat{\Theta}^{\varepsilon}) \approx 9.59$. Note that the level sets indeed resemble those of the first Dirichlet eigenfunction and that $\hat{\Theta}^{\varepsilon}$ behaves like a clipped version of its normal derivative.

Finally, we wish to present numerical results for the minimization problem. As above, we concentrate on the ellipse and the L. With respect to the former we offer in Figures 5 and 6, respectively, the initial iterate supplied to, and final iterate delivered by, the alternating search minimization algorithm presented at the close of the previous section. The domain was approximated by 13374 triangles with $p = 7288$ vertices.

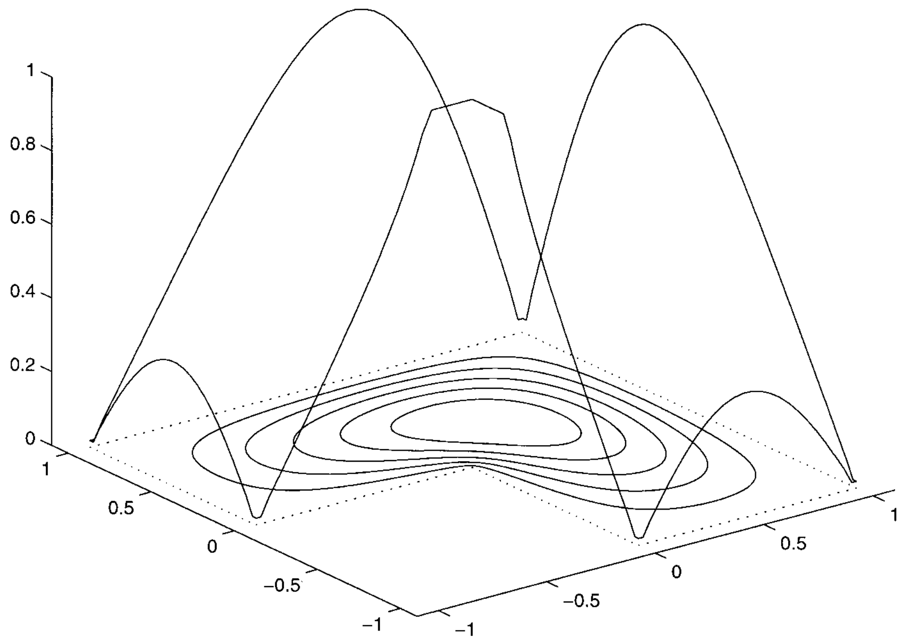

FIG. 4. *Maximal fastening of the L.*

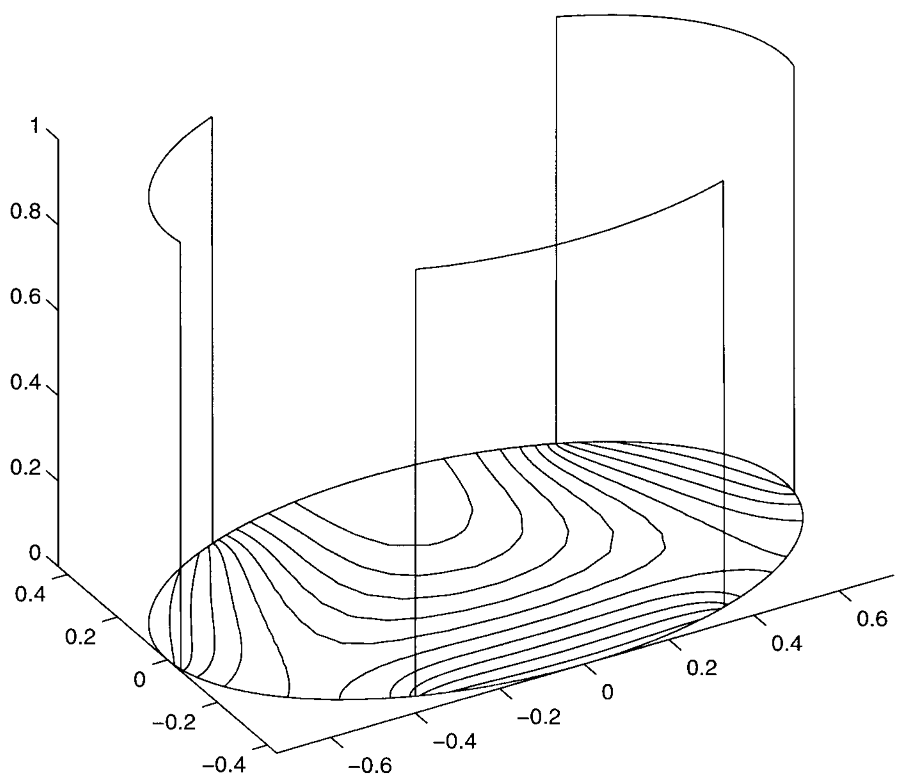

FIG. 5. *Initial iterate.*

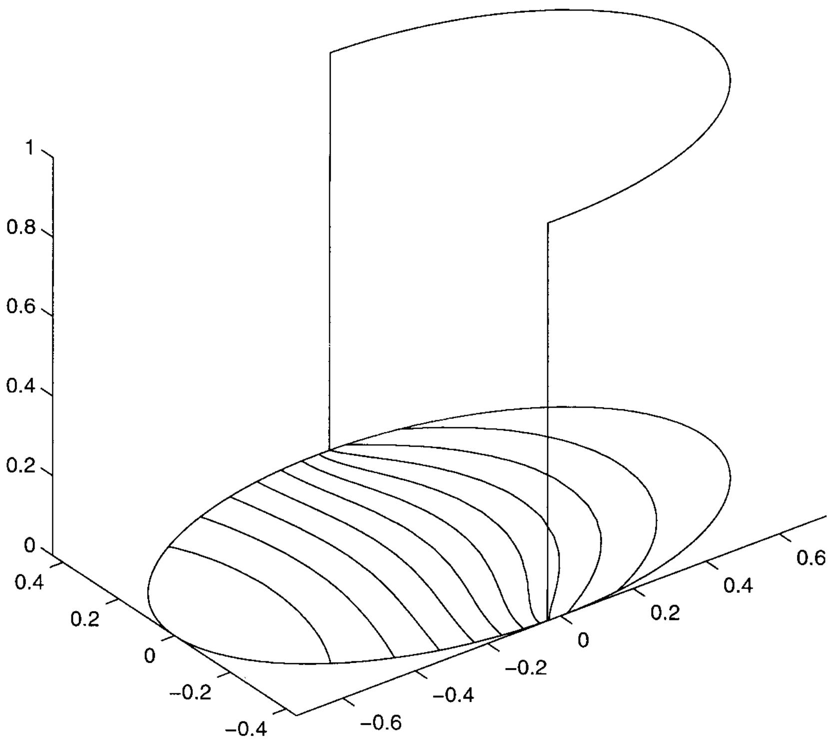

FIG. 6. *Final iterate.*

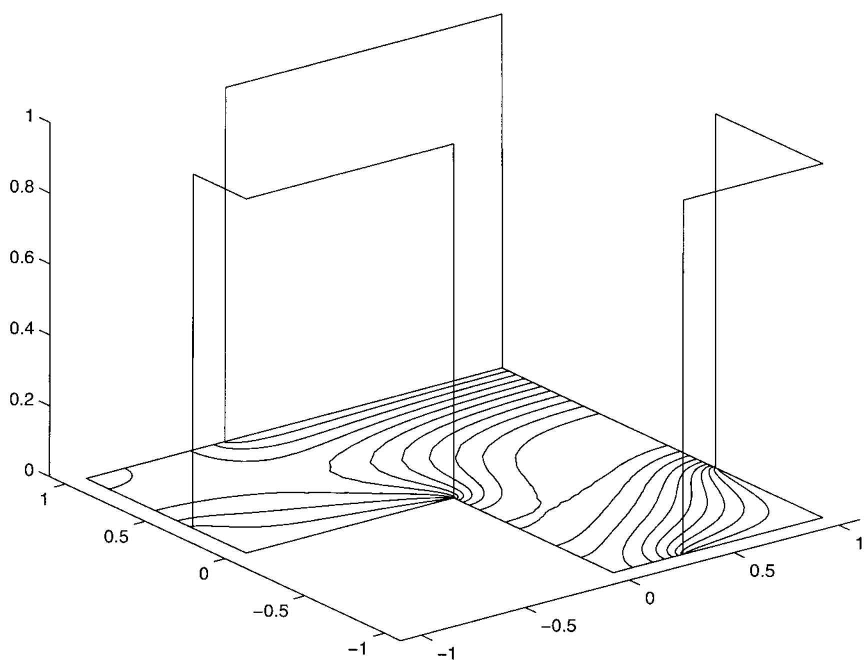

FIG. 7. *Initial iterate.*

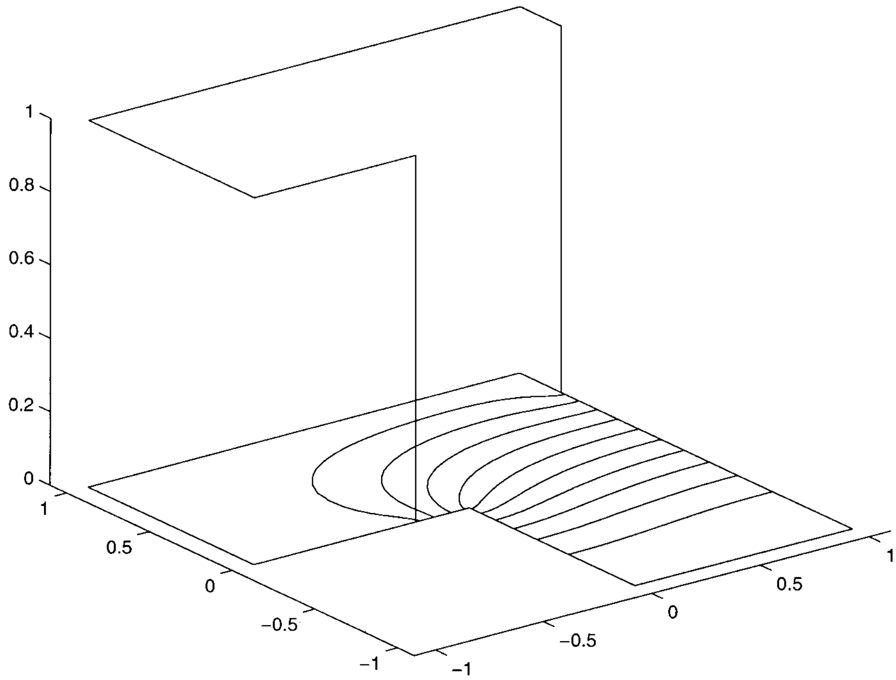

FIG. 8. *Final iterate.*

Its boundary was partitioned into $m = 1200$ edges. With $\gamma = 1/2$ and $\varepsilon = 0.1$ the algorithm came to rest in 69 iterations. The eigenvalue, 6.68, of the initial iterate was diminished to 3.07. In both cases we have also plotted the contours of the associated eigenfunction.

The initial and final iterates, along with the contours of their associated eigenfunctions, for the L-shaped drum are depicted in Figures 7 and 8. In this case the domain was approximated by 18238 triangles with $p = 9936$ edges. Its boundary was partitioned into $m = 1632$ edges. With $\gamma = 1/2$ and $\varepsilon = 0.01$ the algorithm came to rest in 31 iterations and reduced the eigenvalue of the initial iterate, 4.08, to 0.88. We note that the final iterate pulled the Dirichlet data away from the reentrant corner and wrapped it around the outer corner. The resulting eigenvalue is indeed less than 1.09, the eigenvalue of the L with Dirichlet data on the three legs above the diagonal $x = y$.

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