AN INVERSE SPECTRAL PROBLEM FOR A NONNORMAL FIRST ORDER DIFFERENTIAL OPERATOR

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We study the eigenvalues of two restrictions of $B\partial_x + P$ where B is the two–by–two matrix that is zero on the diagonal and one off the diagonal and P is a two–by–two matrix of Lipschitz functions on the unit interval. We establish asymptotic forms for their eigenvalues and associated root vectors and demonstrate that these root vectors constitute a Riesz basis in $L^2(0,1)^2$. We show that our forward analysis makes rigorous the attack on the associated inverse problem by M. Yamamoto, *Inverse spectral problem for systems of ordinary differential equations of first order, I*, J. Fac. Sci. Univ. Tokyo, Sect. 1A, Math. 35, 1988, pp. 519–546. We apply these results to the recovery of the line resistance and leakage conductance of a nonuniform transmission line.

1. **Introduction**

Our intent is to develop a mathematical apparatus with which to recover, from spectral data, dissipative terms in hyperbolic equations. As a model example, let us consider an electrical transmission line composed of two parallel conductors. We let x denote distance along the line and recall that $v(x,t)$, the potential difference between the two conductors, and $i(x,t)$, the current in the "live" conductor, satisfy (see Magnuson et al. [MAT])

$$\begin{aligned} -\ell(x)i_t(x,t) &= v_x(x,t) + r(x)i(x,t) \\ -c(x)v_t(x,t) &= i_x(x,t) + g(x)v(x,t). \end{aligned} \tag{1.1}$$

Here ℓ and c are the respective inductance and capacitance per unit length while r and g signify the respective resistance and leakage conductance per unit length. In the uniform line, where each of these four coefficients is constant, we may differentiate the first equation

† Supported by NSF grant DMS–9258312.

in (1.1) with respect to x and the second with respect to t and so arrive at the classical telegrapher's equation

$$\ell c v_{tt} - v_{xx} + (rc + \ell g)v_t + rgv = 0. \tag{1.2}$$

One recognizes that it is r and g that are responsible for the attenuation in v. We therefore fix $\ell = c = 1$ in (1.1) and attempt to recover nonconstant r and g. The spectral data we speak of will correspond to specific choices of terminal conditions on i and v. In particular, supposing the line to be of unit length, we treat separately the cases

$$v(0) = v(1) = 0 \quad \text{and} \quad v(0) = i(1) = 0.$$

With $u \equiv [i\ v]$ we now write (1.1) as

$$-u_t = \mathcal{A}(P)u$$

where

$$\mathcal{A}(P)u \equiv B\partial_x u + Pu, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad P(x) = \begin{pmatrix} r(x) & 0 \\ 0 & g(x) \end{pmatrix}.$$

In order to accommodate the end conditions we must restrict $\mathcal{A}(P)$ to one of the domains

$$D_k = \{u \in H^1(0,1)^2 : W_0 u(0) + W_k u(1) = 0\}, \quad k = 1, 2$$

where

$$W_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; W_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad W_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We let $\mathcal{A}_k(P)$ denote the restriction of $\mathcal{A}(P)$ to D_k and now investigate the extent to which $\sigma_1(P)$ and $\sigma_2(P)$, the respective spectra of $\mathcal{A}_1(P)$ and $\mathcal{A}_2(P)$, determine P. Assuming that r and g are distinct, we shall see that these two spectra indeed determine P, i.e., *both* r and g.

We shall derive this as a corollary to our main result, Theorem 1.1. This result is stated in terms of a pair, P and Q, of two–by–two (*not* necessarily diagonal) matrices of Lipschitz functions. In terms of three additional functions of P and Q,

$$\theta_1(x) = \tfrac{1}{2}\int_0^x \operatorname{tr} B(P(s) - Q(s))\, ds, \quad \theta_2(x) = \tfrac{1}{2}\int_0^x \operatorname{tr}\,(P(s) - Q(s))\, ds, \quad \text{and}$$

$$R(x) = \mathrm{e}^{\theta_1(x)I + \theta_2(x)B} = \mathrm{e}^{\theta_1(x)} \begin{pmatrix} \cosh\theta_2(x) & \sinh\theta_2(x) \\ \sinh\theta_2(x) & \cosh\theta_2(x) \end{pmatrix}, \tag{1.3}$$

we state

Theorem 1.1. *Suppose that P and Q are two–by–two matrices of Lipschitz functions. $\sigma_1(P)=\sigma_1(Q)$ and $\sigma_2(P)=\sigma_2(Q)$ if and only if*

$$\theta_2(1)=0 \quad \textit{and} \tag{1.4}$$

$$P(x)-BP(x)B=R^{-1}(x)(Q(x)-BQ(x)B)R(x), \quad 0<x<1. \tag{1.5}$$

With respect to the latter condition, (1.5), we note that the trace of each side gives the same value regardless of P and Q and that this is true after left multiplication by B as well. As a result, this condition produces, at most, two independent equations in the elements of P and Q. A concrete manifestation of this remark is

Corollary 1.2. *Suppose P and Q are Lipschitz and that their first rows coincide. If $\sigma_1(P)=\sigma_1(Q)$ and $\sigma_2(P)=\sigma_2(Q)$ then their second rows coincide as well.*

Both the Theorem and its Corollary follow from more general claims of Yamamoto [Y1], [Y2]. Yamamoto however was working under the erroneous assumption that σ_k is composed of purely simple eigenvalues. Note that π is a double eigenvalue of $\mathcal{A}_1(P)$ when

$$P=\begin{pmatrix} 0 & 0 \\ 0 & 2\pi \end{pmatrix}.$$

Moreover, this eigenvalue is defective in the sense that the null space of $\mathcal{A}_1(P)-\pi$ is only one dimensional. Yamamoto's assumption appears to stem from a misreading of Russell's work, [R], on the associated forward problem. In particular, Russell states without proof that the elements of σ_k follow a given asymptotic form and that the corresponding eigenvectors comprise a Riesz basis in $L^2(0,1)^2$.

We offer below a careful proof of Russell's claims. Such results in fact follow almost immediately from the Gelfand–Levitan approach to the inverse problem. Central to this approach is the construction of the transmutation operator X for which $X\mathcal{A}(P)=\mathcal{A}(Q)X$. Though Yamamoto indeed adopted such an approach he failed to recognize the extent to which his results mirrored those previously obtained by Levitan and Sargsjan in their study of the Dirac problem [LS1]. Recall that the Dirac problem is concerned with the operator

$$\mathcal{D}(P)u\equiv J\partial_x u+Pu, \quad J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.6}$$

under the condition on P that $JP+PJ=0$. Although, unlike $\mathcal{A}(P)$, this operator is formally selfadjoint we shall nonetheless see that the transmutation operator constructed by Levitan and Sargsjan for $\mathcal{D}(P)$ leads us immediately to the transmutation operator for $\mathcal{A}(P)$. We believe that this derivation is considerably more transparent than the one offered by Yamamoto. Our derivation is also made less cumbersome by fixing the choice of boundary conditions. It should be mentioned that Yamamoto, on the contrary, uses the two spectra to determine coefficients in the unknown boundary conditions.

Before setting off on the proof of Theorem 1.1 we deduce from it two new corollaries. In the first we use its necessary condition to actually *compute* the eigenvalues of $\mathcal{A}_k(P)$ for a large class of P. As an example, we consider those P that commute with B, i.e., of the form

$$P(x) = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{12}(x) & p_{11}(x) \end{pmatrix}.$$

Corollary 1.3. *If P is Lipschitz and commutes with B then*

$$\sigma_k(P) = \sigma_k(0) + \int_0^1 p_{11}(x)\,dx, \quad k = 1, 2$$

where $\sigma_k(0)$ denotes the spectrum of $B\partial_x$ on D_k.

Proof: If P and Q each commute with B then, as $B^2 = I$, we note that (1.5) is trivially satisfied. In addition, $\theta_2(x) = \int_0^x p_{11} - q_{11}\,ds$, in this case. Upon choosing $Q = I\int_0^1 p_{11}\,dx$ it follows from the Theorem that $\sigma_k(P) = \sigma_k(Q)$. As Q however is just a multiple of the identity we note that $\sigma_k(Q) = \sigma_k(0) + \int_0^1 p_{11}\,dx$. ∎

Regarding the transmission line problem, in the degenerate case that r and g coincide on the entire length of the line it follows from $P = rI$ that

$$\sigma_k(P) = \sigma_k(0) + \int_0^1 r(x)\,dx, \quad k = 1, 2.$$

In the nondegenerate case we find

Corollary 1.4. *Assume that $P = \mathrm{diag}(r_1\ g_1)$ and $Q = \mathrm{diag}(r_2\ g_2)$ are Lipschitz and that $\{x : r_j(x) = g_j(x)\}$ has zero measure for either $j = 1$ or 2. If $\sigma_k(P) = \sigma_k(Q)$, $k = 1, 2$ then $P = Q$.*

Proof: We define $p \equiv r_1 - g_1$ and $q \equiv r_2 - g_2$ and note that (1.5) takes the form

$$p(x)R(x)H = q(x)HR(x) \quad \text{where} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.7}$$

Also, as both P and Q are diagonal, we find that θ_1 is identically zero. Hence

$$R(x) = \begin{pmatrix} \cosh\theta_2(x) & \sinh\theta_2(x) \\ \sinh\theta_2(x) & \cosh\theta_2(x) \end{pmatrix} \quad \text{and} \quad \theta_2(x) = \tfrac{1}{2}\int_0^x (r_1 + g_1 - r_2 - g_2)\,ds$$

Let us now write (1.7) componentwise. Both the diagonal equations read

$$p(x)\cosh\theta_2(x) = q(x)\cosh\theta_2(x) \quad \text{and hence} \quad p(x) = q(x), \tag{1.8}$$

while the offdiagonal equations state

$$p(x)\sinh\theta_2(x) = -q(x)\sinh\theta_2(x) \quad \text{and so} \quad p(x) = -q(x) \text{ or } \theta_2(x) = 0. \tag{1.9}$$

From $p = q$ comes the fact that $r_1 - r_2 = g_1 - g_2$ and so

$$\theta_2(x) = \int_0^x (g_1 - g_2)\,ds. \tag{1.10}$$

Regarding the first choice in (1.9) we note that $p = -q$ now implies that $r_1 = g_1$ *and* $r_2 = g_2$. As, by hypothesis, this cannot occur on a set of positive measure it follows that θ_2 must vanish. From (1.10) we now deduce that $g_1 = g_2$ everywhere and hence that $r_1 = r_2$ there as well. ∎

The remainder is organized as follows. In §2 we show that $\sigma_k(P)$ is indeed comprised of an infinite sequence of eigenvalues and that the associated root vectors are complete in $L^2(0,1)^2$. In §3 we derive the aforementioned transmutation operator. We apply this operator in §4 in deriving asymptotic forms for the eigenvalues and root vectors and so conclude that the root vectors comprise a Riesz basis for $L^2(0,1)^2$. In §5 we finally prove Theorem 1.1. Note that this proof requires only the completeness of the root vectors. We include the Riesz basis result because it follows easily from the eigenvalue asymptotics and because it *is* needed in our subsequent work, [CK]. We close in §6 with comments on the stability question and a numerical reconstruction algorithm.

2. Existence of Eigenvalues and Completeness of Root Vectors

Under the assumption that P is bounded we characterize the eigenvalues of $\mathcal{A}_1(P)$ as poles of its resolvent, $(\mathcal{A}_1(P)-\lambda)^{-1}$, and equivalently, as zeros of a so–called "shooting" function. The analysis of $\mathcal{A}_2(P)$ requires no additional effort. To compute the resolvent of $\mathcal{A}_1(P)$ is to solve

$$Bv' + Pv - \lambda v = f, \quad f \in L^2(0,1)^2, \quad v \in D_1,$$

or, upon multiplying through by B,

$$v' = B(\lambda - P)v + Bf. \tag{2.1}$$

Its solution, in terms of the solution of

$$Y'(x,\lambda) = B(\lambda - P)Y(x,\lambda), \quad Y(0,\lambda) = I,$$

is simply

$$v(x,\lambda) = Y(x,\lambda)\eta + Y(x,\lambda)\int_0^x Y^{-1}(t,\lambda)Bf(t)\,dt,$$

where η is determined by $W_0 v(0,\lambda) + W_1 v(1,\lambda) = 0$, i.e., by

$$(W_0 + W_1 Y(1,\lambda))\eta + W_1 Y(1,\lambda)\int_0^1 Y^{-1}(t,\lambda)Bf(t)\,dt = 0.$$

As

$$W_0 + W_1 Y(1,\lambda) = \begin{pmatrix} 0 & 1 \\ Y_{21}(1,\lambda) & Y_{22}(1,\lambda) \end{pmatrix} \tag{2.2}$$

we find that (2.1) has the solution

$$\begin{aligned} v(x,\lambda) &= \int_0^1 Y(x,\lambda)\{\chi_{[0,x]}(t) - \\ &\qquad (W_0 + W_1 Y(1,\lambda))^{-1} W_1 Y(1,\lambda)\} Y^{-1}(t,\lambda)Bf(t)\,dt \\ &= (\mathcal{A}_1(P) - \lambda)^{-1} f, \end{aligned}$$

so long as $Y_{21}(1,\lambda) \neq 0$. Hence, the eigenvalues of $\mathcal{A}_1(P)$ are the zeros of $\lambda \mapsto Y_{21}(1,\lambda)$. The order of such a zero is its order as a pole of the resolvent and therefore is the algebraic multiplicity of the eigenvalue. The geometric multiplicity of an eigenvalue is the dimension of its corresponding eigenspace. We may now prove

Theorem 2.1. *If P is bounded then $\sigma_1(P)$ consists of an infinite number of geometrically simple eigenvalues of finite algebraic multiplicity, $\{\lambda_n\}_n$, that cluster nowhere in the finite plane. The corresponding root vectors are complete in $L^2(0,1)^2$. In addition, the eigenvalues satisfy*

$$|\Re\lambda_n| \leq \tfrac{1}{2}\|P + P^T\|_\infty.$$

Proof: We write $\mathcal{A}_1$ as a bounded perturbation of a real skewadjoint operator with a Hilbert-Schmidt resolvent and invoke a standard result of Goberg and Krein [GK]. In particular, $\mathcal{A}_1 = \mathcal{A}_0 + P$ where $\mathcal{A}_0 = B\partial_x$. With f and g in D_1 we find

$$\langle \mathcal{A}_0 f, g\rangle = \int_0^1 f_2'\overline{g}_1 + f_1'\overline{g}_2\,dx = -\int_0^1 f_2\overline{g}_1' + f_1\overline{g}_2'\,dx = -\langle f, \mathcal{A}_0 g\rangle.$$

That is, $\mathcal{A}_0$ is skewadjoint. As such, $1 \notin \sigma(\mathcal{A}_0)$ and so

$$(\mathcal{A}_0 - 1)^{-1} f = \int_0^1 \mathrm{e}^{Bx}\{\chi_{[0,x]}(t) - (W_0 + W_1\mathrm{e}^B)^{-1}W_1\mathrm{e}^B\}\mathrm{e}^{-Bt}Bf(t)\,dt$$

This is Hilbert–Schmidt, $P(\mathcal{A}_0-1)^{-1}$ is compact, and so $(\mathcal{A}_0-1)^{-1}P(\mathcal{A}_0-1)^{-1}$ is Hilbert–Schmidt. Having satisfied the hypotheses of [GK, Theorem V.10.1], it follows that $\sigma_1(P)$ is indeed comprised of an infinite sequence, without a finite limit point, of eigenvalues of finite algebraic multiplicity and that the associated sequence of root vectors is complete in $L^2(0,1)^2$.

Regarding the geometric multiplicity, if $\lambda \in \sigma_1(P)$ then each eigenvector is of the form $u(x) = Y(x,\lambda)\eta$ where η lies in the null space of $W_0 + W_1 Y(1,\lambda)$. As the rank of $W_0 + W_1 Y(1,\lambda)$ never dips below one, see (2.2), it follows that u is determined up to a multiplicative constant.

The bound on the real part comes from taking the inner product of

$$Bu' + Pu = \lambda u$$

with u. One finds,

$$2i\Im \int_0^1 u_2' \overline{u}_1 \, dx + \langle Pu, u\rangle = \lambda,$$

i.e., $\Re\lambda = \Re\langle Pu,u\rangle = \frac{1}{2}\langle (P+P^T)u,u\rangle$. ∎

3. The Transmutation Operator

With respect to the Dirac problem Levitan and Sargsjan construct a transmutation operator of the form, see [LS1, (IX.3.6)],

$$X(P,Q)u(x) \equiv R(x)u(x) + \int_0^x K(x,s)u(s)\,ds. \tag{3.1}$$

On $D_0 \equiv \{u \in H^1(0,1)^2 : u_2(0) = 0\}$ this form suffices to transmute $\mathcal{A}$ as well.

Theorem 3.1. *If R is given by (1.3) and K satisfies the partial differential equation*

$$B\partial_x K(x,y) + \partial_y K(x,y)B = K(x,y)P(y) - Q(x)K(x,y) \tag{3.2}$$

for $(x,y) \in \Omega \equiv \{(s,t) : 0 < t < s < 1\}$, and the boundary conditions

$$K(x,x)B - BK(x,x) = BR'(x) + Q(x)R(x) - R(x)P(x) \tag{3.3}$$

$$K_{12}(x,0) = K_{22}(x,0) = 0, \tag{3.4}$$

for $0 \le x \le 1$, then, as operators on D_0,

$$X(P,Q)\mathcal{A}(P) = \mathcal{A}(Q)X(P,Q). \tag{3.5}$$

Proof: Given $u \in D_0$ we compute

$$\begin{aligned}\mathcal{A}(Q)X(P,Q)u = BR(x)u'(x) + \{BR'(x) + Q(x)R(x) + BK(x,x)\}u(x) \\ + \int_0^x \{BD_xK(x,s) + Q(x)K(x,s)\}u(s)\,ds\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}X(P,Q)\mathcal{A}(P)u = R(x)Bu'(x) + \{R(x)P(x) + K(x,x)B\}u(x) \\ - K(x,0)Bu(0) + \int_0^x \{K(x,s)P(s) - D_sK(x,s)B\}u(s)\,ds.\end{aligned} \tag{3.7}$$

Equating the coefficients of $u'(x)$ in (3.6) and (3.7) we find $BR(x) = R(x)B$ and so

$$R(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & a(x) \end{pmatrix}. \tag{3.8}$$

Equating like terms in $u(x)$ brings

$$BR'(x) + Q(x)R(x) - R(x)P(x) = K(x,x)B - BK(x,x). \tag{3.9}$$

Note that both $KB - BK$ and $BKB - K$ have zero trace. In terms of the left hand side of (3.9) this remark leads to the following linear system for a and b,

$$\begin{pmatrix} a \\ b \end{pmatrix}' = \begin{pmatrix} \mathrm{tr}\,B(P-Q) & \mathrm{tr}\,(P-Q) \\ \mathrm{tr}\,(P-Q) & \mathrm{tr}\,B(P-Q) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.10}$$

Multiplying the first by a and the second by b and subtracting gives

$$(a^2 - b^2)' = (a^2 - b^2)\mathrm{tr}\,B(P-Q).$$

Hence $a^2 - b^2 = \exp(2\theta_1)$. On substituting $a = \exp(\theta_1)\cosh\eta$ and $b = \exp(\theta_1)\sinh\eta$ one finds $\eta = \theta_2$ and so R is given by (1.3).

Now equating the integrands in (3.6) and (3.7) we find that the kernel K satisfies (3.2). One interprets (3.9) as a boundary condition on K, hence (3.3). The remaining boundary condition stems from the $u(0)$ term in (3.7). In particular, $K(x,0)Bu(0) = 0$ and $u_2(0) = 0$ together give (3.4). ∎

Of course one must show that (3.2)–(3.4) has a solution. Levitan and Sargsjan approach the corresponding problem in the Dirac case by solving (3.2) subject to the (unknown) Cauchy data

$$K(x,0) = \begin{pmatrix} \alpha(x) & 0 \\ \beta(x) & 0 \end{pmatrix}.$$

Inserting this solution into (3.3) is then said to determine α and β. Neither in [LS1], nor in their updated account [LS2], however, do they provide a careful accounting of this approach. In particular, they fail to specify the necessary hypotheses on P and Q.

Yamamoto [Y1] takes a different approach and provides considerably more detail. Let us briefly review his argument. He notes that (3.2), in terms of the four–vector L, where

$$\begin{aligned} L_1 &= K_{12} - K_{21}, \quad L_2 = K_{11} - K_{22}, \\ L_3 &= K_{11} + K_{22}, \quad L_4 = K_{12} + K_{21}, \end{aligned} \tag{3.11}$$

takes the especially simple form

$$(\partial_x + \Gamma \partial_y) L(x,y) = F(x,y) L(x,y), \tag{3.12}$$

where $\Gamma = \mathrm{diag}(-1,-1,1,1)$ and $F(x,y)$ is a four-by-four matrix composed of linear combinations of elements of $P(y)$ and $Q(x)$. In addition, the boundary conditions on K take the form

$$L_k(x,x) = r_k(x), \quad k = 1,2, \tag{3.13}$$

$$L_3(x,0) = L_2(x,0) \quad \text{and} \quad L_4(x,0) = -L_1(x,0), \tag{3.14}$$

where r_k is the $(1,k)$ element of $BR'(x) + Q(x)R(x) - R(x)P(x)$. Via the method of successive approximations applied to an equivalent integral equation formulation Yamamoto proves that (3.12)–(3.14) possesses a unique C^1 solution when both P and Q are C^1. As the uniform convergence of Yamamoto's approximant's requires only uniform bounds on P and Q and their derivatives it follows in fact that (3.12)–(3.14) possesses a unique Lipschitz solution when both P and Q are Lipschitz. Solving for K in terms of L in (3.11) gives

Theorem 3.2. *If P and Q are Lipschitz then* (3.2)–(3.4) *possesses a unique Lipschitz solution.*

The idea behind the proof of Theorem 1.1 is to use the spectral data to determine the value of K at $x = 1$ and to exploit the sense in which this information is propagated to the line $y = x$. Of particular interest is the case when K is assumed to vanish when $x = 1$. Yamamoto has shown that K must be identically zero in such a case. Extending his result to the Lipschitz case we find

Theorem 3.3. *If K satisfies* (3.2)–(3.4) *and*

$$K(1,y) = 0, \quad 0 \le y \le 1,$$

then K is identically zero in Ω. In particular,

$$P(x) - BP(x)B = R^{-1}(x)(Q(x) - BQ(x)B)R(x), \quad 0 \le x \le 1.$$

Proof: For the first claim see [Y1, Lemma 2]. Regarding the second we note that if K vanishes then, recalling (3.3), so to must $BR' + QR - RP$. Using the fact that $BR = RB$ and $R(Q + BQB) = (Q + BQB)R$ we note that

$$BR' = R(\theta_1' B + \theta_2' I) = \tfrac{1}{2}R(P + BPB - Q - BQB) = \tfrac{1}{2}R(P + BPB) - \tfrac{1}{2}(Q + BQB)R.$$

The claim is now a direct consequence of $BR' + QR - RP = 0$. ∎

Recognizing the latter condition as (1.5) It remains to show that spectral data indeed determines K at $x = 1$ and to demonstrate the relevance of the trace condition, (1.4). We shall address the latter in the next section while the former requires knowledge of the fact that our transmutation operator maps root spaces of $\mathcal{A}_k(P)$ onto root spaces of $\mathcal{A}_k(Q)$. As preparation we deduce an easy corollary of Theorem 3.1.

Theorem 3.4. *If P and Q are Lipschitz and u satisfies*

$$(\mathcal{A}(P) - \lambda)u = f, \quad u(0) = \xi$$

then $v \equiv X(P,Q)u$ and $g \equiv X(P,Q)f$ satisfy

$$(\mathcal{A}(Q) - \lambda)v = g, \quad v(0) = \xi. \tag{3.15}$$

Proof: From $R(0) = I$ follows $v(0) = u(0)$. In addition,

$$(\mathcal{A}(Q) - \lambda)v = (\mathcal{A}(Q) - \lambda)X(P,Q)u = X(P,Q)(\mathcal{A}(P) - \lambda)u = X(P,Q)f = g. \ ∎$$

We now show when $\sigma_k(P) = \sigma_k(Q)$ that $X(P,Q)$ maps root spaces onto root spaces. This requires some notation. The root space corresponding to an eigenvalue $\lambda_0 \in \sigma_k(P)$ of algebraic multiplicity m is denoted

$$\mathcal{L}(\mathcal{A}_k(P), \lambda_0) \equiv \{u \in D_k : (\mathcal{A}_k(P) - \lambda_0)^m u = 0\}.$$

Corollary 3.5. *If $\sigma_k(P) = \sigma_k(Q)$ then $\mathcal{L}(\mathcal{A}_k(Q), \lambda_0) = X(P,Q)\mathcal{L}(\mathcal{A}_k(P), \lambda_0)$.*

Proof: We choose as basis for $\mathcal{L}(\mathcal{A}_k(P), \lambda_0)$ the set $\{u_j\}_{j=0}^{m-1}$ where

$$\begin{aligned} &(\mathcal{A}_k(P) - \lambda_0)u_0 = 0, \\ &(\mathcal{A}_k(P) - \lambda_0)u_j = u_{j-1}, \quad \langle u_j, u_0 \rangle = 0, \quad j = 1, \ldots, m-1. \end{aligned} \tag{3.16}$$

Let $v_0 = X(P,Q)u_0$. By the previous Theorem we find that $(\mathcal{A}(Q) - \lambda_0)v_0 = 0$ and $v_0(0) = u_0(0)$. As $\lambda_0 \in \sigma_k(Q)$ and solutions to the initial value problem are unique it follows that $(\mathcal{A}_k(Q) - \lambda_0)v_0 = 0$.

Next, Let $v_1 = X(P,Q)u_1$. By the previous Theorem we find that $(\mathcal{A}(Q) - \lambda_0)v_1 = v_0$ and $v_1(0) = u_1(0)$. As above, it follows that $(\mathcal{A}_k(Q) - \lambda_0)v_1 = v_0$. One proceeds with $v_j = X(P,Q)u_j$ for the remaining j with impunity. ∎

4. Asymptotics

Given a Lipschitz Q we denote the elements of $\sigma_1(Q)$ by $\{\lambda_n(Q)\}_n$. In this section we show that

$$\lambda_n(Q) = \tfrac{1}{2}\int_0^1 \operatorname{tr} Q\, dx + in\pi + O(1/|n|) \tag{4.1}$$

and we prove that the associated root vectors comprise a Riesz basis for $L^2(0,1)^2$. These results follow from applying Theorem 3.1 with $P \equiv 0$, $f \equiv 0$, and $\xi = [1,0]$. In this case,

$$u(x,\lambda) = [\cosh \lambda x, \sinh \lambda x]$$

while

$$\begin{aligned} v(x,\lambda) &= R(x)u(x,\lambda) + \int_0^x K(x,s)u(s,\lambda)\, ds \\ &= R(x)u(x,\lambda) - \frac{1}{\lambda}\int_0^x K(x,s)B\partial_s u(s,\lambda)\, ds. \end{aligned}$$

As K is Lipschitz this last equality will permit us to estimate v in terms of

$$\kappa \equiv \max_{0\le x\le 1} \int_0^x |\partial_s K(x,s)|\, ds.$$

Theorem 4.1. *If Q is Lipschitz then there exists a finite C for which*

$$\left| v(x,\lambda) - e^{\theta_1(x)} \begin{pmatrix} \cosh(\lambda x + \theta_2(x)) \\ \sinh(\lambda x + \theta_2(x)) \end{pmatrix} \right| \le \frac{C}{|\lambda|}$$

componentwise, uniformly in x for $|\Re\lambda| \le \|Q\|_\infty$.

Proof: On the left is simply $|v - Ru|$ so it remains only to estimate, via integration by parts,

$$\begin{aligned} \left|\int_0^x K(x,s)B\partial_s u(s,\lambda)\, ds\right| &= \Big| K(x,0)u(0,\lambda) - K(x,x)u(x,\lambda) \\ &\qquad + \int_0^x \partial_s K(x,s)Bu(s,\lambda)\, ds \Big| \\ &\le (2\|K\|_\infty + \kappa)\|u(\cdot,\lambda)\|_\infty \\ &\le (2\|K\|_\infty + \kappa)e^{2\|Q\|_\infty}. \end{aligned}$$ ∎

As $\{\lambda_n(Q)\}_n$ is nothing but the set of zeros of $\lambda \mapsto v_2(1,\lambda)$ we may now expect the $\lambda_n(Q)$ to lie asymptotically close to the zeros of of $\mathrm{e}^{\theta_1(1)}\sinh(\lambda+\theta_2(1))$, i.e., to

$$\nu_n \equiv \tfrac{1}{2}\int_0^1 \mathrm{tr}\, Q(x)\,dx + in\pi, \quad n = \pm 1, \pm 2, \ldots$$

Such a result follows, as usual, from Rouche's Theorem once it is established that

$$\frac{C\mathrm{e}^{-\theta_1(1)}}{|\lambda|} < |\sinh(\lambda+\theta_2(1))| \tag{4.2}$$

for λ restricted to a circle of radius $O(1/|n|)$ centered about ν_n. This inequality is proven by Cox and Zuazua in [CZ, Lemma 5.2] on the circles

$$\begin{aligned}\Gamma_N &\equiv \{z : |z+\theta_2(1)| = N\pi + \pi/2\}, \quad \text{and}\\ \Gamma_{\pm n} &\equiv \{z : |z+\theta_2(1) \mp in\pi| = 2C\mathrm{e}^{-\theta_1(1)}/(n\pi)\}, \quad n > N,\end{aligned}$$

where N is the least integer greater than $4C\mathrm{e}^{-\theta_1(1)}/\pi$. From this it follows that

Theorem 4.2. *If Q is Lipschitz then $\mathcal{A}_1(Q)$ has exactly $2N$ eigenvalues, including multiplicity, in Γ_N and one simple eigenvalue in $\Gamma_{\pm n}$ for each $n > N$. This exhausts the spectrum of $\mathcal{A}_1(Q)$.*

Of course, formula (4.1) is an immediate corollary of this result. In proving that the associated root vectors indeed form a Riesz basis we continue to follow [CZ]. In particular, as a consequence of Bari's Theorem, see [CZ, Theorem 6.1], we need only

(i) construct a sequence, $\{\tilde{v}_n\}_{n>N}$ of eigenvectors of $\mathcal{A}_1(Q)$ corresponding to $\{\lambda_n(Q)\}_{n>N}$
(ii) construct a sequence, $\{\tilde{w}_n\}_{n>N}$ of eigenvectors of the adjoint $\mathcal{A}_1^*(Q)$ corresponding to $\{\overline{\lambda}_n(Q)\}_{n>N}$
(iii) establish that $\langle \tilde{v}_j, \tilde{w}_k\rangle = \delta_{jk}$ and
(iv) demonstrate that both $\{\langle \tilde{v}_n, [f,g]\rangle\}_{n>N}$ and $\{\langle \tilde{w}_n, [f,g]\rangle\}_{n>N}$ lie in ℓ^2 for each $[f,g] \in L^2(0,1)^2$.

With respect to (i) we note that the eigenvalue asymptotics allow us to improve on the basic estimate of Theorem 4.1. In particular, up to a scalar multiple, the eigenvector of $\mathcal{A}_1(Q)$ corresponding to $\lambda_n(Q)$ must be of the form

$$\begin{aligned} v_n(x) \equiv v(x,\lambda_n(Q)) &= \mathrm{e}^{\theta_1(x)}\begin{pmatrix}\cosh(\nu_n x + \theta_2(x) + O(1/|n|)x)\\ \sinh(\nu_n x + \theta_2(x) + O(1/|n|)x)\end{pmatrix} + O(1/|\lambda_n|)\\ &= \mathrm{e}^{\theta_1(x)}\begin{pmatrix}\cosh(\xi(x) + in\pi x)\\ \sinh(\xi(x)+in\pi x)\end{pmatrix} + O(1/|n|)\end{aligned}$$

where

$$2\xi(x) = x\int_0^1 \operatorname{tr} Q\, ds - \int_0^x \operatorname{tr} Q\, ds$$

measures the deviation of $\operatorname{tr} Q$ from constant.

We may argue similarly that the eigenvector w_n of $\mathcal{A}_1^*(Q)$ associated with $\overline{\lambda}_n$ behaves, up to a scalar multiple, like

$$w_n(x) = \mathrm{e}^{\theta_1(x)} \begin{pmatrix} \cosh(\xi(x) - in\pi x) \\ -\sinh(\xi(x) - in\pi x) \end{pmatrix} + O(1/|n|).$$

Now, if j and k are distinct and greater than N then $\langle v_j, w_k\rangle = 0$ for strictly algebraic reasons. In particular,

$$\lambda_j\langle v_j, w_k\rangle = \langle \mathcal{A}_1(Q)v_j, w_k\rangle = \langle v_j, \mathcal{A}_1^*(Q)w_k\rangle = \lambda_k\langle v_j, w_k\rangle.$$

That the two sequences may be binormalized follows from their asymptotic forms. In particular,

$$\langle v_n, w_n\rangle = \int_0^1 \mathrm{e}^{2\theta_1(x)}\, dx + O(1/|n|),$$

and so

$$\begin{aligned} \tilde{v}_n &\equiv \frac{v_n}{\sqrt{\langle v_n, w_n\rangle}} = \frac{v_n}{\|\mathrm{e}^{\theta_1}\|_2} + O(1/n) \quad \text{and} \\ \tilde{w}_n &\equiv \frac{w_n}{\sqrt{\langle v_n, w_n\rangle}} = \frac{w_n}{\|\mathrm{e}^{\theta_1}\|_2} + O(1/n) \end{aligned}$$

together satisfy condition (iii). We now check that $\{\langle \tilde{v}_n, [f,g]\rangle\}$ is square summable,

$$\begin{aligned} \langle \tilde{v}_n, [f,g]\rangle &= \frac{1}{\|\mathrm{e}^{\theta_1}\|_2}\int_0^1 \{\cosh(\xi(x) + in\pi x)\overline{f}(x) + \\ &\qquad \sinh(\xi(x) + in\pi x)\overline{g}(x)\}\mathrm{e}^{\theta_1(x)}\, dx + O(1/n) \\ &= \int_0^1 \{\phi(x)\cos(n\pi x) + i\psi(x)\sin(n\pi x)\}\, dx + O(1/n) \end{aligned}$$

where

$$\begin{aligned} \phi(x) &= (\overline{f}(x)\cosh\xi(x) + \overline{g}(x)\sinh\xi(x))\mathrm{e}^{\theta_1(x)}/\|\mathrm{e}^{\theta_1}\|_2 \quad \text{and} \\ \psi(x) &= (\overline{f}(x)\sinh\xi(x) + \overline{g}(x)\cosh\xi(x))\mathrm{e}^{\theta_1(x)}/\|\mathrm{e}^{\theta_1}\|_2. \end{aligned}$$

As ξ is bounded we find that both ϕ and ψ lie in $L^2(0,1)$ and so $\{\langle \tilde{v}_n, [f,g]\rangle\} \in \ell^2$. The argument with respect to $\tilde{w}_n$ follows exactly as above. As the full set of root vectors associated with $\mathcal{A}_1(Q)$ have already been shown to be complete, recall Theorem 2.1, the Theorem of Bari may now be applied.

Theorem 4.3. *If Q is Lipschitz then the root vectors associated with $\mathcal{A}_1(Q)$ constitute a Riesz basis in $L^2(0,1)^2$.*

5. Proof of Theorem 1.1

Although the proof differs little from that found in [Y1] we find it sufficiently short to warrant its inclusion on the grounds of completeness. We label $\sigma_1(P)$ and $\sigma_1(Q)$ in terms of $\{\lambda_n(P)\}_n$ and $\{\lambda_n(Q)\}_n$ and $\sigma_2(P)$ and $\sigma_2(Q)$ by $\{\mu_n(P)\}_n$ and $\{\mu_n(Q)\}_n$.

(Necessity) If (1.5) holds then $K \equiv 0$ and so $X(P,Q)u(x) = R(x)u(x)$. If, in addition, (1.4) is satisfied then $R(1) = \mathrm{e}^{\theta_1(1)}I$. From Theorem 3.4 it now follows that if $u(x,\lambda_n(P))$ is a root vector of $\mathcal{A}_1(P)$ corresponding to $\lambda_n(P)$ then $R(x)u(x,\lambda_n(P))$ is a root vector of $\mathcal{A}_1(Q)$ corresponding to $\lambda_n(P)$. As $R(x)$ is clearly invertible it follows that $\sigma_1(P) = \sigma_1(Q)$. Arguing in an identical fashion one finds $\sigma_2(P) = \sigma_2(Q)$.

(Sufficiency) If the spectra coincide then (1.4) is surely satisfied and so $R(1) = \mathrm{e}^{\theta_1(1)}I$ as above. In light of Theorem 3.3, to conclude that (1.5) is satisfied we need only show that K vanishes on the line $x = 1$.

As $R(1)$ is a multiple of the identity and $u_2(1,\lambda_n(P)) = 0$ it follows that the second component of $R(1)u(1,\lambda_n(P))$ vanishes. As $\{\lambda_n(P)\}_n = \{\lambda_n(Q)\}_n$ it follows that the second component of $X(P,Q)u(1,\lambda_n(P))$ vanishes as well. As a result, the second component of (3.1) when $x = 1$ and $u = u(x,\lambda_n(P))$ reads simply

$$0 = \int_0^1 \{K_{21}(1,s)u_1(s,\lambda_n(P)) + K_{22}(1,s)u_2(s,\lambda_n(P))\}\,ds.$$

From the completeness of $\{u(x,\lambda_n(P))\}_n$ in $L^2(0,1)^2$ it now follows that the second column of K vanishes when $x = 1$.

Similarly, if $u(x,\mu_n(P))$ denotes a root vector of $\mathcal{A}_2(P)$ corresponding to $\mu_n(P)$ then the first component of $R(1)u(1,\mu_n(P))$ vanishes. If $\{\mu_n(P)\}_n = \{\mu_n(Q)\}_n$ then the first component of $X(P,Q)u(1,\mu_n(P))$ vanishes as well. Hence, the first component of (3.1) when $x = 1$ and $u = u(x,\mu_n(P))$ states

$$0 = \int_0^1 \{K_{11}(1,s)u_1(s,\mu_n(P)) + K_{12}(1,s)u_2(s,\mu_n(P))\}\,ds.$$

From the completeness of $\{u(x,\mu_n(Q))\}_n$ in $L^2(0,1)^2$ it follows that the first column of K also vanishes when $x = 1$. This concludes the proof of Theorem 1.1. For the proof of Corollary 1.2 see [Y3, Corollary 1].

6. **Stability and Reconstruction**

In [Y2] Yamamoto addresses the stability question that issues from Corollary 1.3. In particular, he claims that if (i) P and Q have identical first rows and (ii) $\sigma_1(P)$ is close to $\sigma_1(Q)$ and $\sigma_2(P)$ is close to $\sigma_2(Q)$, then the first row of P is close to the second row of Q. As in [Y1] this claim is proven under the false assumption that all eigenvalues are simple. When such is the case one can indeed establish the lemma that underlies Yamamoto's stability result. Namely, the root vectors of $\mathcal{A}_k(P)$ vary continuously with P. The necessary repairs to Yamamoto's stability result fit more naturally in the context of [CK]. For, with stability, comes a license to search out a numerical method for the reconstruction of P from $\sigma_1(P)$ and $\sigma_2(P)$. In particular, extending the techniques of Rundell and Sacks [RS] we reconstruct, in [CK], two coefficients in the damped wave equation

$$\rho(x)u_{tt}(x,t) - (a(x)u_x(x,t))_x + d(x)u_t(x,t) = 0,$$

given one of the coefficients and two spectra of

$$\begin{pmatrix} 0 & I \\ \rho^{-1}\partial_x a \partial_x & -d/\rho \end{pmatrix}. \tag{6.1}$$

The spectra of (6.1) correspond to the spectra of certain quadratic pencils, see [CZ], as well as to spectra of first order systems, see [Y3]. In this context, our results correspond to the infinite dimensional analog of the problem treated by Veselić in [V1] and [V2].

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AMS subject classifications: 34A55, 34L10, 34L20