Aye, There's the Rub,
An inquiry in to why a plucked string comes to rest

Steven J. Cox

1. **Introduction**

Newton's first law implies that a plucked string will remain in motion unless impeded by some additional force. I wish to identify the nature of this dissipative force. I take the broad view – To identify is to construct a mathematical model that yields a reasonable fit to experimental data.

Our first task will be to gather experimental data. In §2 we measure the time and frequency response of a plucked string. We build a preliminary model in §3 via the Principle of Least Action. We augment this in §4 arriving at the *damped wave equation*. We determine, numerically, that the dissipative force that best matches experimental data is concentrated at the strings ends, where it rubs against its supports. As further demonstration, we place a magnetic damper at the string's midpoint and show that our technique detects both its strength and position.

2. **Acquiring the Data**

Regarding equipment, I have followed, to a large extent, the advice of Lord Rayleigh [R, Vol.1, p. §125]

> "For quantitative investigations into the laws of strings, the sonometer is employed. By means of a weight hanging over a pulley, a catgut, or a metallic wire, is stretched across two bridges mounted on a resonance case. A movable bridge, whose position is estimated by a scale running parallel to the wire, gives the means of shortening the efficient portion of the wire to any desired extent. The vibrations may be excited by plucking, as in the harp, or with a bow (well supplied with rosin), as in the violin."

Rayleigh proceeded to estimate the string's natural frequencies by aural comparison with struck tuning forks of known natural frequency. I adopt the sonometer but substitute for ear and fork an electromagnetic pick–up, analog–to–digital converter, and the Discrete Fourier Transform. For hardware, with regard to the photograph in Figure 2.1, I have used the WA–9611 Sonometer (long object in foreground) and WA–9613 Detector Coil (small black box positioned under the midpoint of the string) of PASCO Scientific `www.pasco.com` and an AT–MIO–16E Data Acquisition Card from National Instruments `www.natinst.com`. The detector coil returns a voltage that is proportional to the time rate of change of the coil's magnetic flux which in turn is proportional to the velocity of the string in a neighborhood of the detector.

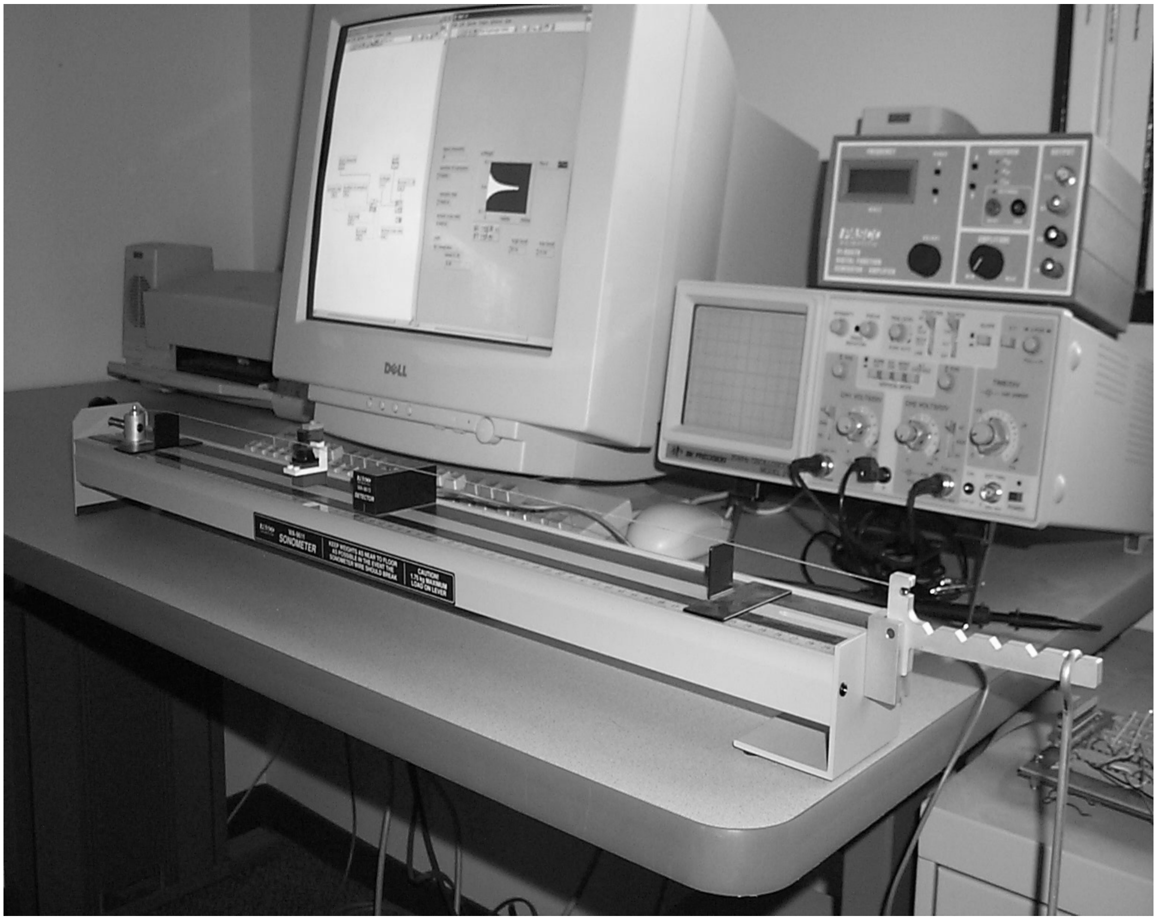

Figure 2.1. The Experimental Apparatus.

The chosen string possessed a uniform linear density of

$$\rho = 0.0015\, kg/m. \tag{2.1}$$

The distance between the two black posts provided an effective length of

$$\ell = 0.6\, m. \tag{2.2}$$

Finally, suspending 0.55 kg from the lever in the right corner, produced a tension of

$$\tau = 26.95\, m\, kg/s^2. \tag{2.3}$$

The small Lego object to the left of the detector is the magnetic damper that I used in the experiment of §6. In a typical experiment I would pluck the string, in the absence of the magnetic damper, and sample the voltage, 10000 times per second, for about 15 seconds. One would need a page at least 18 inches wide in order to produce a meaningful plot of such a time series. We content ourselves therefore with a plot of every tenth sample.

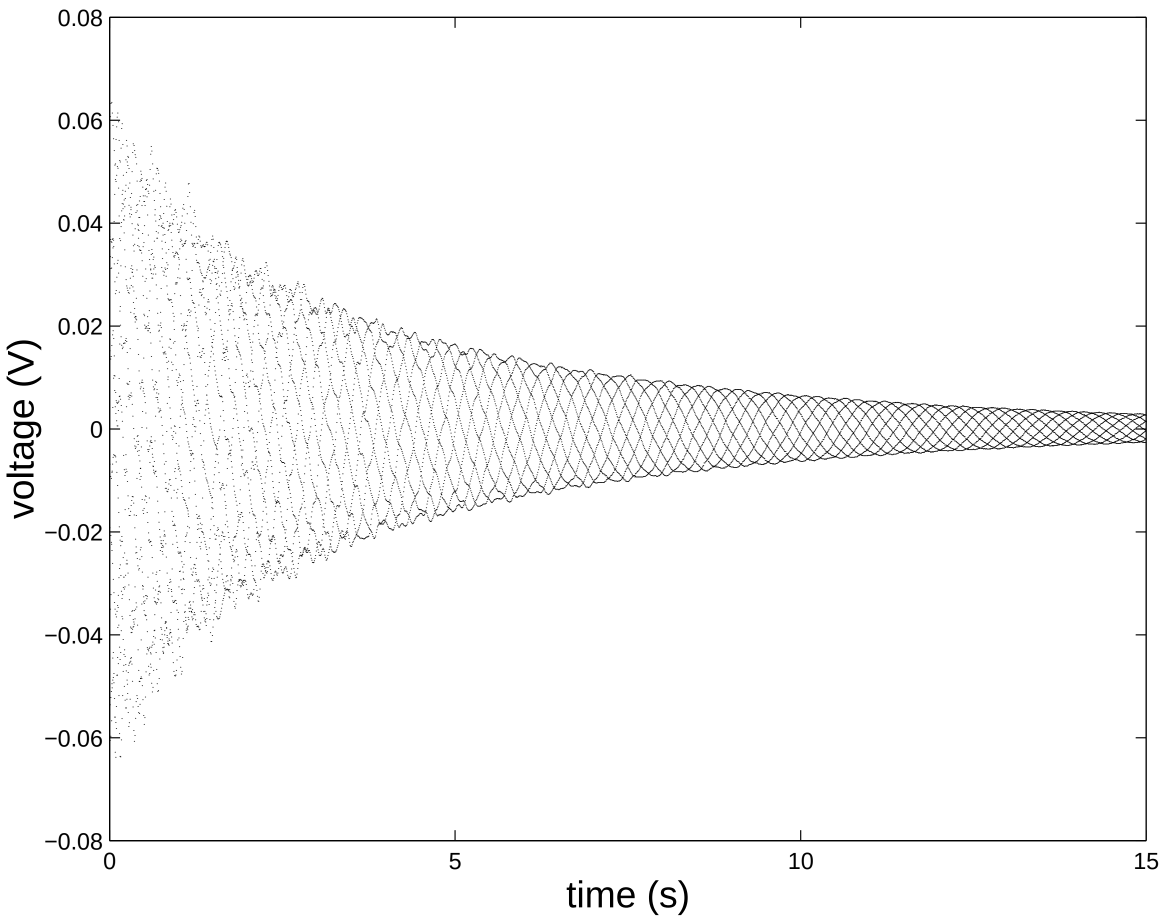

Figure 2.2. Response of a plucked string.

This signal appears, at least eventually, to be composed of a sum of decaying sinusoids. The Discrete Fourier Transform is an excellent tool for extracting their respective frequencies. If v denotes the vector of $N = 150000$ voltages depicted in Figure 2.2 then the Discrete Fourier Transform of v is

$$v_k^{\#} = \sum_{n=1}^{N} v_n \exp(-2\pi i(k-1)(n-1)/N), \quad 1 \le k \le N.$$

The relationship between $v^{\#}$ and the Fourier coefficients a and b in

$$v(n) = a_0 + \sum_{k=1}^{N/2} a_k \cos\left(2\pi \frac{k}{Nh} t(n)\right) + b_k \sin\left(2\pi \frac{k}{Nh} t(n)\right) \tag{2.4}$$

is

$$a_0 = \frac{1}{N} v_1^{\#}, \quad a_k = \frac{2}{N} \Re(v_{k+1}^{\#}), \quad \text{and} \quad b_k = -\frac{2}{N} \Im(v_{k+1}^{\#}),$$

where $t(n)$ is the time at the nth sample and $h = 0.0001$ is the time (in seconds) between samples. We have used $\Re$ and $\Im$ to denote the real and imaginary parts respectively. In light of (2.4) we plot the natural logarithm of the magnitude of $v_k^{\#}$ (computed via the `fft` command in Matlab) versus the frequency

$$\omega_k = \frac{k}{Nh}.$$

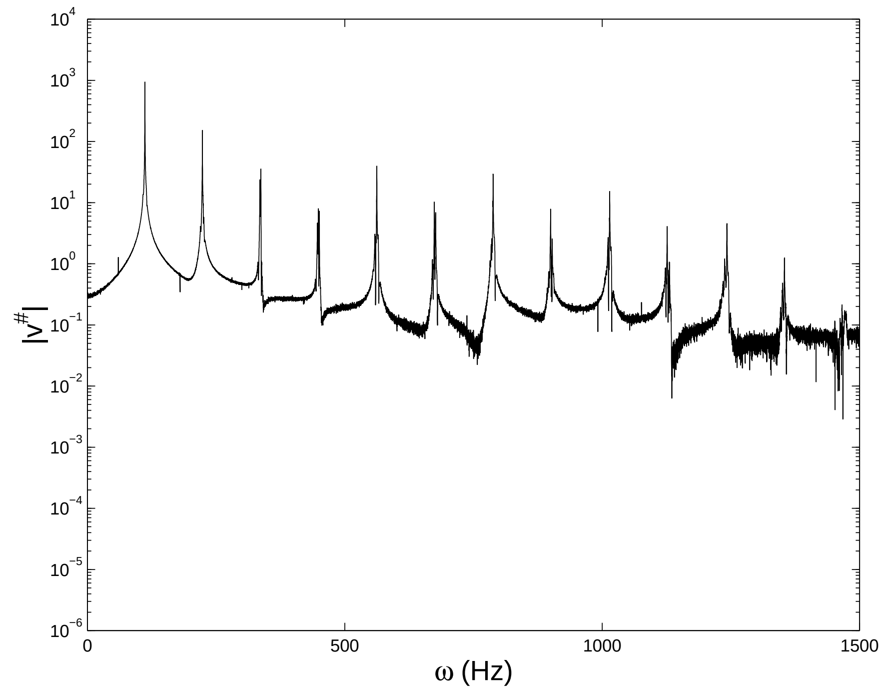

Figure 2.3. Magnitude of the Discrete Fourier Transform of Figure 2.2

One sees immediately that the energy in v is concentrated at integer multiples of about 111 Hz. We speak of these as the resonant (or natural) frequencies of the string. In order to discern at what rate(s) the associated resonant modes decay we attempt to fit our voltage readings to a function of the form

$$\phi(t;p) = \sum_{j=1}^{m} p_{j,1} \exp(p_{j,2}t) \cos(2\pi p_{j,3}t + p_{j,4}) \tag{2.5}$$

parametrized by the m-by-4 matrix p. By 'fit' we mean to choose p in order that the sum, over all samples, of the squares of the differences is minimized. That is we choose the p that solves

$$\min_{p\in\mathbf{R}^{4m}} \sum_{n=1}^{N} |v_n - \phi(nh;p)|^2 \tag{2.6}$$

This problem is readily solved in Matlab with a call to `leastsq`. In particular, asking for

$m = 8$ 4–tuples, Matlab returned

$$p = \begin{pmatrix} -0.0485 & -0.2181 & 111.6934 & -3.5923 \\ -0.0149 & -0.4956 & 223.3987 & -2.5630 \\ -0.0038 & -0.7865 & 335.1198 & -1.5076 \\ -0.0010 & -1.0670 & 446.8129 & -0.2750 \\ 0.0050 & -0.5977 & 562.1369 & -1.8865 \\ 0.0019 & -0.9026 & 673.8576 & -0.8854 \\ 0.0050 & -0.8177 & 788.0784 & -1.1569 \\ 0.0017 & -1.0837 & 899.7653 & 0.0524 \end{pmatrix} \qquad (2.7)$$

and a value of 0.333 for (2.6). We recognize in the third column of p the aforementioned resonant frequencies. The second column lists their respective decay rates. We wish to point out that the higher frequencies decay 4 to 5 times faster than the lowest. Although p has indeed captured the correct resonant frequencies you may wonder what the associated ϕ looks like. Is 0.333 a good fit when $N = 150,000$ samples are used? We have plotted this ϕ in Figure 2.4. For ease of comparison with Figure 2.2 we have again presented only every tenth sample.

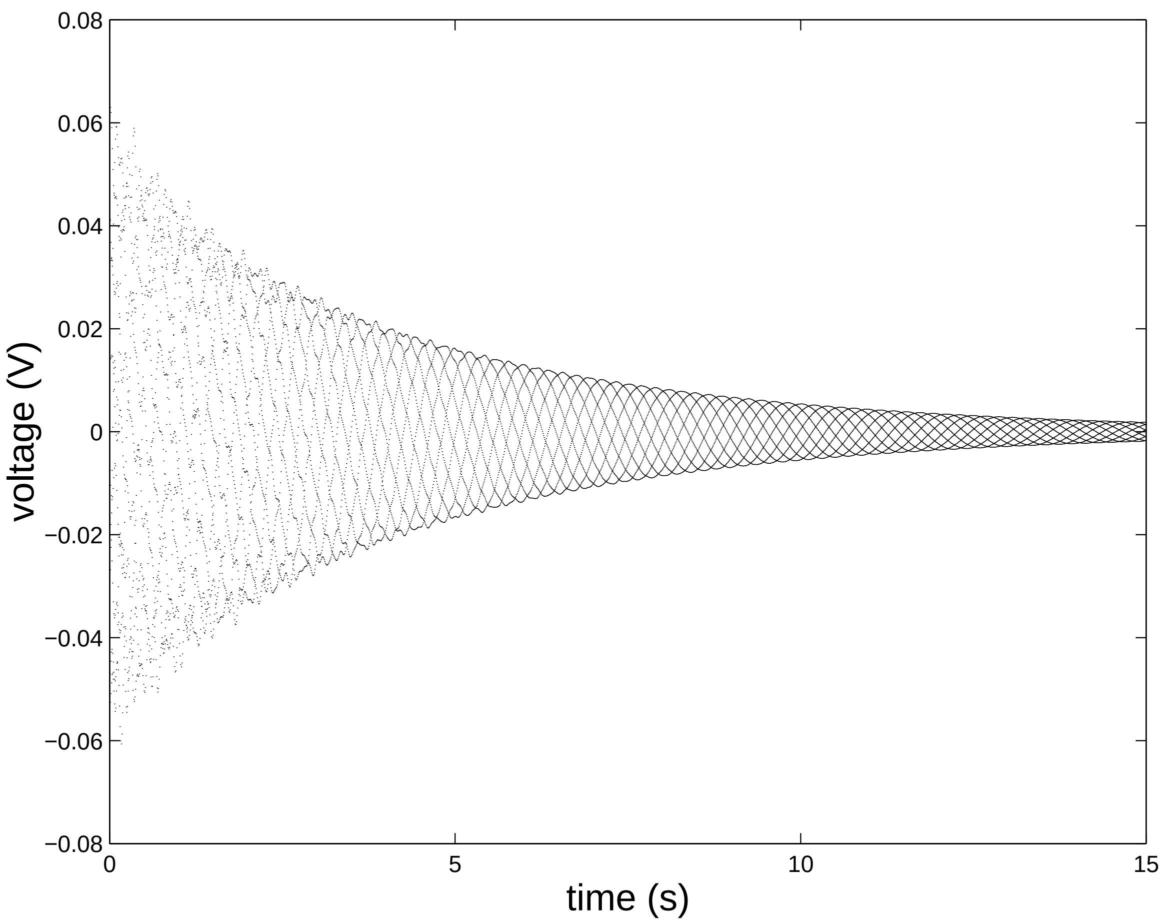

Figure 2.4. Best fit, in the sense of (2.6), to Figure 2.2.

Content that we are on the right track we shall use p as the descriptor of our string.

Our goal is now to devise a mathematical model that, at least, predicts behavior in line with the second and third columns of p.

3. A Mathematical Model

Implicit in all of the above is the assumption that the string's motion is planar and purely transverse to its rest state. Leaving this unchallenged (for the moment) we denote by ℓ the distance between the string's two supports and denote a material point by $(x,0)$ where $x \in [0,\ell]$. At time t this material point lies at the point $(x, u(x,t))$. Assuming the string to be taut we expect zero displacement at its two ends, i.e.,

$$u(0,t) = u(\ell,t) = 0, \qquad \forall\, t \geq 0. \tag{3.1}$$

In addition, we suppose that the pluck uniquely specifies both the position and velocity at each point in x at the initial instant of time. In symbols this means

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = v_0(x) \tag{3.2}$$

where u_0 and v_0 are known. Our task is to find and analyze the equation satisfied by $u(x,t)$ for $x \in (0,\ell)$ and all future time. Of the many possible paths to the string equation the most venerable is the one originating in the the Calculus of Variations. The guiding principle goes variously under the names *Principle of Least Action*, *Principle of Stationary Action*, and *Hamilton's Principle*. Though already known to Leibniz and Euler the pronouncements of Maupertuis in 1746 in his "The laws of motion and of rest deduced from a metaphysical principle," brought it to life. In the words of Maupertuis,

> "If there occurs some change in nature, the amount of action necessary for this change must be as small as possible."

Of course in order to actually apply such a principle one must first quantify this mysterious 'action.' In the field of rigid–body mechanics, see, e.g., Arnold [A, p. 59] or Gelfand and Fomin [GF, §21], the action is the time integral of the difference of the kinetic and potential energies. The latter text, in §36, extends these notions to continua, such as our string. Let us take a look.

Roughly speaking, the string's kinetic energy is half the product of its mass and the square of its velocity. Its mass is simply the product of its density, ρ, and length, ℓ. As $u(x,t)$ is the height of point x at time t, the velocity of the point is simply the time rate of change of u at x, i.e., $u_t(x,t)$. Summing over all points of the string we posit a kinetic energy of the form

$$T(u,t) \equiv \tfrac{1}{2}\rho \int_0^\ell u_t^2(x,t)\,dx.$$

Next, the potential energy is the work (product of force and distance) required to deform the string. The force in our context is the tension, τ, while the distance is simply the difference in lengths between the deformed and undeformed states. As the deformed state is assumed to be a graph we arrive at a potential energy of the form

$$U(u,t) = \tau \left\{ \int_0^\ell \sqrt{1 + u_x^2(x,t)}\,dx - \ell \right\}. \tag{3.3}$$

Left in place, the square root in U would clutter our application of the Principle of Least Action. Are there grounds for replacing the square root with something more agreeable? Even very strong plucks of the 60 *cm* string on the sonometer of figure 1 produced maximal displacements of less than 1 *cm*. This suggests that 1/60 is a reasonable upper bound for $|u_x(x,0)|$. As time increases the pluck travels along the string and then smoothes out and dissipates. That is, $|u_x(x,t)|$ is not likely to exceed 1/60 for any $t \ge 0$. As a result, we may suppose that $u_x^2(x,t) < 1/3600$. As this is considerably smaller than 1 we may use $\sqrt{1+z} \approx 1 + z/2$ in (3.3) to arrive at

$$U(u,t) = \tfrac{1}{2}\tau \int_0^\ell u_x^2(x,t)\,dx. \tag{3.4}$$

From T and U we now assemble the action

$$A(u) \equiv \int_{t_0}^{t_1} \{T(u,t) - U(u,t)\}\,dt,$$

over the time window $[t_0, t_1]$. The Principle of Least Action now asserts that the *actual* motion, u, is the one that minimizes A among all *possible* motions. The weaker statement that A is stationary at u will finally provide us an equation for u. In order to make this precise let us compute the derivative A at u in the direction v,

$$\langle A'(u), v\rangle \equiv \lim_{h\to 0} \frac{A(u+hv) - A(u)}{h}.$$

The direction v is chosen so that $u + hv$ is indeed a possible motion. By that we mean that $u+hv$ and u should satisfy the same boundary conditions with respect to both space and time. This of course means that

$$v(0,t) = v(\ell,t) = 0 \qquad \forall\, t > 0 \quad \text{and} \quad v(x,t_0) = v(x,t_1) = 0 \qquad \forall\, x \in (0,\ell). \tag{3.5}$$

Now

$$\begin{aligned} A(u+hv) - A(u) &= \tfrac{1}{2} \int_{t_0}^{t_1} \int_0^\ell (\rho\{(u_t + hv_t)^2 - u_t^2\} - \tau\{(u_x + hv_x)^2 - u_x^2\})\,dx\,dt \\ &= h \int_{t_0}^{t_1} \int_0^\ell (\rho u_t v_t - \tau u_x v_x)\,dx\,dt + \tfrac{1}{2}h^2 \int_{t_0}^{t_1} \int_0^\ell (\rho v_t^2 - \tau v_x^2)\,dx\,dt \end{aligned}$$

and so after dividing by h and letting h tend to zero we find

$$\langle A'(u), v\rangle = \int_{t_0}^{t_1} \int_0^\ell (\rho u_t v_t - \tau u_x v_x)\,dx\,dt. \tag{3.6}$$

Now if u is indeed a stationary point of A then (3.6) should vanish for every possible direction v. The consequences of this vanishing would be easier to read off if v rather than its derivatives were to appear in (3.6). To that end let us note that

$$u_t v_t = (u_t v)_t - u_{tt} v$$

and so, recalling (3.5),

$$\begin{aligned}\int_{t_0}^{t_1} u_t(x,t)v_t(x,t)\,dt &= \int_{t_0}^{t_1}((u_t(x,t)v(x,t))_t - u_{tt}(x,t)v(x,t))\,dt \\ &= u_t(x,t)v(x,t)\Big|_{t=t_0}^{t=t_1} - \int_{t_0}^{t_1} u_{tt}(x,t)v(x,t)\,dt \\ &= -\int_{t_0}^{t_1} u_{tt}(x,t)v(x,t)\,dt. \end{aligned} \tag{3.7}$$

Similarly

$$\int_0^\ell u_x(x,t)v_x(x,t)\,dx = -\int_0^\ell u_{xx}(x,t)v(x,t)\,dx. \tag{3.8}$$

On substitution of (3.7) and (3.8) into (3.6) we arrive at

$$\langle A'(u),v\rangle = \int_{t_0}^{t_1}\int_0^\ell (\rho u_{tt}(x,t) - \tau u_{xx}(x,t))v(x,t)\,dx\,dt. \tag{3.9}$$

If this indeed vanishes for every v on $(0,\ell)\times(t_0,t_1)$ satisfying (3.5) then necessarily the coefficient of v must vanish identically. This implication is typically called the *Fundamental Lemma of the Calculus of Variations*. Its application to (3.9) yields the so called string equation,

$$\rho u_{tt}(x,t) - \tau u_{xx}(x,t) = 0 \quad \text{in} \quad (0,\ell)\times(t_0,t_1). \tag{3.10}$$

This linear partial differential equation possesses an infinite number of independent solutions, namely

$$u(x,t) = f(x + t\sqrt{\tau/\rho}) + g(x - t\sqrt{\tau/\rho})$$

where f and g are *arbitrary* twice differentiable functions. As f and g describe *waves* with *speed* $\sqrt{\tau/\rho}$ one often speaks of (3.10) as the **wave equation**. One of course arrives at a unique solution to (3.10) by asking u to obey the boundary, (3.1), and initial, (3.2), conditions. In the absence of boundary conditions the reader may wish to check that it is a simple matter to express f and g in terms of u_0 and v_0. Boundaries produce reflections and therefore a more complicated representation for u. Our initial concern however is not with the *exact* values of u but rather with the question of how, or indeed whether, it decays.

One typically measures decay by studying the, so called, instantaneous total energy

$$E(t) \equiv T(u,t) + U(u,t) = \int_0^\ell \rho u_t^2 + \tau u_x^2\,dx. \tag{3.11}$$

Now by decay we mean that E should be decreasing. On evaluating its time derivative we find however that

$$E'(t) = 2\int_0^\ell \rho u_{tt}u_t + \tau u_{xt}u_x\,dx = 2\int_0^\ell (\rho u_{tt} - \tau u_{xx})u_t\,dx = 0.$$

That is, the energy in the initial pluck is conserved throughout time. This of course damns (3.10) as a model of what we observed in figure 2. Recalling however that that signal was essentially a linear combination of damped sinusoids we might ask whether or not the string equation at least gets the frequencies right. This will require us to actually solve (3.10).

4. **Solving the Wave equation**

By analogy to solving linear second order ordinary differential equations we write (3.10) and (3.2) as the first order system

$$V_t = \mathcal{A}V, \qquad V(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \tag{4.1}$$

where

$$V \equiv \begin{pmatrix} u \\ u_t \end{pmatrix} \quad \text{and} \quad \mathcal{A} \equiv \begin{pmatrix} 0 & I \\ \frac{\tau}{\rho}\frac{d^2}{dx^2} & 0 \end{pmatrix}$$

and u of course is still assumed to satisfy the boundary conditions (3.1). This $\mathcal{A}$ is a matrix differential operator that acts on a subspace of $\mathcal{E}$, the Hilbert space of vectors V with finite energy. The inner product on $\mathcal{E}$ is the one that naturally stems from E, namely, for arbitrary V and W in $\mathcal{E}$,

$$\langle V, W\rangle \equiv \int_0^\ell \{\tau\partial_x V_1 \partial_x \overline{W}_1 + \rho V_2 \overline{W}_2\}\, dx. \tag{4.2}$$

By natural we mean only that the square of the associated norm, $\sqrt{\langle V, V\rangle}$, is precisely the energy of V.

As in the ordinary case one solves (4.1) in terms of the eigenvectors of $\mathcal{A}$, i.e., the solutions of

$$\mathcal{A}\Phi = \lambda\Phi.$$

It is not difficult to recognize these in the doubly infinite sequences

$$\Phi_{\pm n} = \sin(n\pi x/\ell) \begin{pmatrix} \frac{1}{n\pi}\sqrt{\frac{\ell}{\tau}} \\ \pm i\frac{1}{\sqrt{\ell\rho}} \end{pmatrix} \quad \text{and} \quad \lambda_{\pm n} = \pm\frac{in\pi}{\ell}\sqrt{\frac{\tau}{\rho}} \qquad n = 1, 2, \ldots \tag{4.3}$$

We have, for convenience, chosen $\Phi_{\pm n}$ to have unit norm in $\mathcal{E}$. On substituting the values of ρ, τ and ℓ associated with the string of §2 we find the eigenvalues

$$\lambda_{\pm n} = \pm in(701.83) = \pm i2\pi n(111.7).$$

These are in remarkable agreement with the resonant frequencies displayed in Figure 2.3.

As the $\Phi_{\pm n}$ constitute an orthonormal basis in $\mathcal{E}$ we find that the full solution to (4.1) takes the form

$$V(t) = \sum_{n=1}^{\infty} \gamma_{\pm n} \exp(\lambda_{\pm n} t)\Phi_{\pm n}(x) \tag{4.4}$$

where the $\gamma_{\pm n}$ are simply inner products of the $\Phi_{\pm n}$ with the initial pluck, i.e.,

$$\gamma_{\pm n} = \langle V(0), \Phi_{\pm n} \rangle = \sqrt{\frac{\tau}{\ell}} \int_0^\ell u_x(x,0) \cos(n\pi x/\ell)\, dx \mp i \sqrt{\frac{\rho}{\ell}} \int_0^\ell u_t(x,0) \sin(n\pi x/\ell)\, dx.$$

Although the V of (4.4) is composed of terms that oscillate (at the right frequencies), none of them exhibit any decay.

5. The Damped Wave Equation

Of the many ways that one may elicit decay perhaps the simplest, by way of the analogy of adding a dashpot to a mass–spring system, is to introduce into the wave equation a term that is proportional to velocity. That is, to consider

$$\rho u_{tt} - \tau u_{xx} + 2au_t = 0. \tag{5.1}$$

for some constant a, in units of $kg/m/s$. Retaining our same notion of total energy we ask whether E now decreases when u satisfies (5.1) rather than (3.10). Note that its time derivative

$$E'(t) = 2 \int_0^\ell (\rho u_{tt} - \tau u_{xx}) u_t \, dx = -4a \int_0^\ell u_t^2 \, dx$$

is negative so long as $a > 0$. That is, positive a produces decay of energy, E. So good so far, let us determine the effect of a on the eigenvalues and eigenvectors of the previous section.

With regard to the associated first order system, (4.1), the $\mathcal{A}$ operator now takes the form

$$\mathcal{A}(a) = \begin{pmatrix} 0 & I \\ \frac{\tau}{\rho}\frac{d^2}{dx^2} & \frac{-2a}{\rho} \end{pmatrix}$$

As a is constant the eigenvectors of $\mathcal{A}(a)$ are exactly as in (4.3). The eigenvalues become however

$$\lambda_{\pm n} = -a/\rho \pm \sqrt{(a/\rho)^2 - (n\pi/\ell)^2 \tau/\rho} \tag{5.2}$$

and hence, so long as

$$a < \frac{\pi}{\ell}\sqrt{\frac{\tau}{\rho}}\rho \approx 1.057 \; \frac{kg}{m\, s}$$

the real part of each eigenvalue is $-a/\rho$. The solution to (4.1) with $\mathcal{A}$ now replaced by $\mathcal{A}(a)$ remains (4.4) with the λ now given by (5.2). Although this indeed produces decay it produces it in a far to uniform fashion. More precisely, each term decays at precisely the same rate, namely a/rho. If we trust the larger variations in decay rates reported in the second column of p then we should search for a generalization of (5.1) that can exhibit such variable rates of decay. We now wish to argue that t suffices to let a vary with x.

When a varies with x, although expansions like (4.4) are still valid, we no longer have explicit expressions for the eigenfunctions and eigenvalues. For that reason we turn to their numerical approximation. More precisely, we solve

$$\mathcal{A}(a)\Psi = \Lambda\Psi \tag{5.3}$$

by supposing Ψ to be a linear combination of the first $2m$ modes of the undamped problem. That is, we suppose

$$\Psi = \sum_{k=1}^{m} (\gamma_k \Phi_k + \gamma_{-k}\Phi_{-k}) \tag{5.4}$$

where the $\Phi_{\pm k}$ are specified in (4.3). On substituting (5.4) into (5.3) and taking the inner product of each side with one of these low, undamped modes we arrive at

$$\langle \mathcal{A}(a)\Psi, \Phi_j\rangle = \Lambda\langle \Psi, \Phi_j\rangle, \quad j = 1, \ldots, m, -1, \ldots, -m \tag{5.5}$$

A moment's reflection now permits us to see in this the matrix eigenvalue problem

$$G(a)\Gamma = \Lambda\Gamma \tag{5.6}$$

for

$$\Gamma = (\gamma_1\ \gamma_2\ \ldots\ \gamma_m\ \gamma_{-1}\ \gamma_{-2}\ \ldots\ \gamma_{-m})^T$$

where $G(a)$ is the $2m$-by-$2m$ matrix with elements

$$G_{j,k}(a) = \begin{cases} \langle \mathcal{A}(a)\Phi_k, \Phi_j\rangle & \text{if } j \le m,\ k \le m \\ \langle \mathcal{A}(a)\Phi_k, \Phi_{m-j}\rangle & \text{if } j > m,\ k \le m \\ \langle \mathcal{A}(a)\Phi_{m-k}, \Phi_j\rangle & \text{if } j \le m,\ k > m \\ \langle \mathcal{A}(a)\Phi_{m-k}, \Phi_{m-j}\rangle & \text{if } j > m,\ k > m. \end{cases} \tag{5.7}$$

The inner product remains the one defined in (4.2). I hope that this notation does not obscure the relatively simple computations required to assemble $G(a)$. Perhaps we should write out in full what is going on in say the first line of (5.7). If $1 \le j \le m$ and $1 \le k \le m$ then

$$\mathcal{A}(a)\Phi_k = \lambda_k\Phi_k + i\frac{1}{\rho\sqrt{\ell\rho}}\begin{pmatrix} 0 \\ -2a(x)\sin(k\pi x/\ell)\end{pmatrix}$$

and so

$$\begin{aligned}\langle \mathcal{A}(a)\Phi_k, \Phi_j\rangle &= \lambda_k\langle \Phi_k, \Phi_j\rangle - 2\frac{1}{\ell\rho}\int_0^\ell a(x)\sin(k\pi x/\ell)\sin(j\pi x/\ell)\,dx \\ &= \frac{ik\pi}{\ell}\sqrt{\frac{\tau}{\rho}}\delta_{j,k} - 2\frac{1}{\ell\rho}\int_0^\ell a(x)\sin(k\pi x/\ell)\sin(j\pi x/\ell)\,dx,\end{aligned}$$

where $\delta_{j,k}$ is zero unless $j = k$ in which case it is one. So, at bottom, $G(a)$ is composed of integrals of the damping against products of sine functions. Upon constructing $G(a)$ we may use the `eig` routine in Matlab and arrive at the $2m$ matrix eigenvalues, $\{\Lambda_{\pm j}(a)\}_{j=1}^m$. As $G(a)$ is real these eigenvalues appear in complex conjugate pairs. Hence, there is no loss in ordering them according to their imaginary parts, i.e.,

$$0 < \Im\Lambda_1(a) < \Im\Lambda_2(a) < \cdots < \Im\Lambda_m(a), \quad \Lambda_{-j} = \overline{\Lambda_j}.$$

Our goal now is to produce an a such that these matrix eigenvalues fit the second two columns of p, namely the complex vector

$$p_j^* = p_{j,2} + i2\pi p_{j,3}.$$

In the interest of practicality we limit our search to a finite dimensional class of dampings. In keeping with the above choice of basis we suppose that

$$a(x) = \sum_{k=1}^{n} \alpha_k \sin((2k-1)\pi x/\ell).$$

Our limitation to odd sine terms reflects our belief that the 'proper' a ought to be symmetric with respect to the midpoint, $x = \ell/2$. Finally, to fit the $\Lambda_j(a)$ to the p_j^* is to solve

$$\min_{\alpha \in \mathbf{R}^n} \sum_{j=1}^{m} |p_j^* - \Lambda_j(a)|^2. \tag{5.8}$$

Of course, as the p_j^* constitute only 16 real parameters we may only hope to find a unique minimizer when $n \le 16$. Asking for the most, we take $n = 16$ and solve (5.8) in Matlab via `leastsq`. We have plotted the resulting a in the figure below.

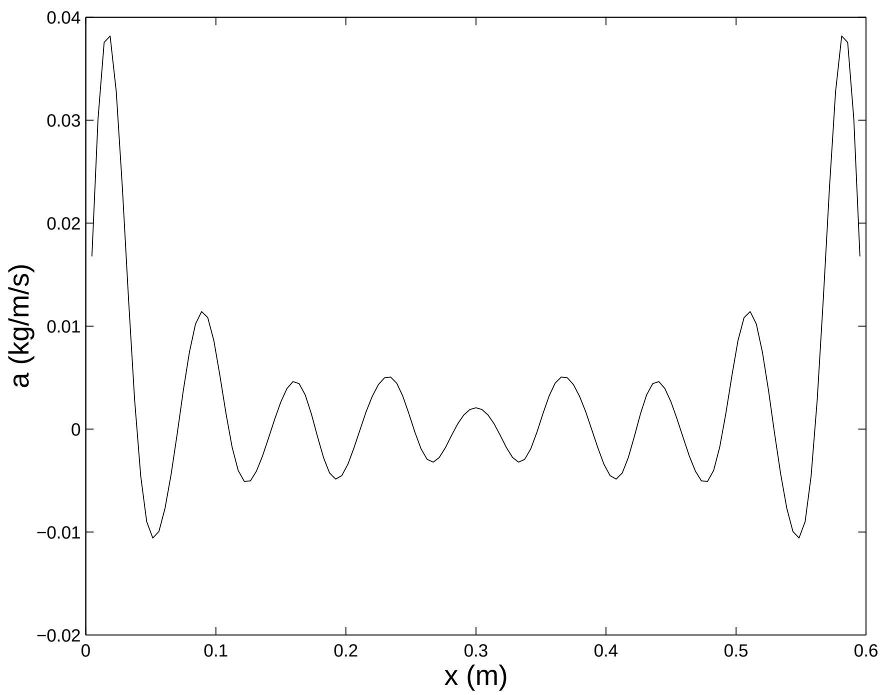

Figure 5.1. The rub.

This result indeed exposes the sought after rub. Namely, a is large at the the two ends, signifying that through rubbing against its supports the string eventually sheds the energy delivered by the pluck.

Of course one should not accept the veracity of a model based on the result of a single experiment. Rather than simply replucking the same string however we wish to see whether our methods might be able to distinguish the presence of an artificial damper.

6. **Discerning the Presence of Additional Damping**

It is not unnatural to attempt to identify the magnitude and position of dissipative mechanisms. As a simple example, engineers seek means by which the size and location of a leak in a cable may be discerned from measurements taken near the cable's ends. We therefore add a dissipative mechanism, a pair of attracting magnets at the center the string (recall Figure 2.1), and attempt to detect it by the eigenvalue matching method used above.

On placing the detector 15 *cm* to the right of the magnetic damper we pluck the string and record the data plotted in Figure 6.1.

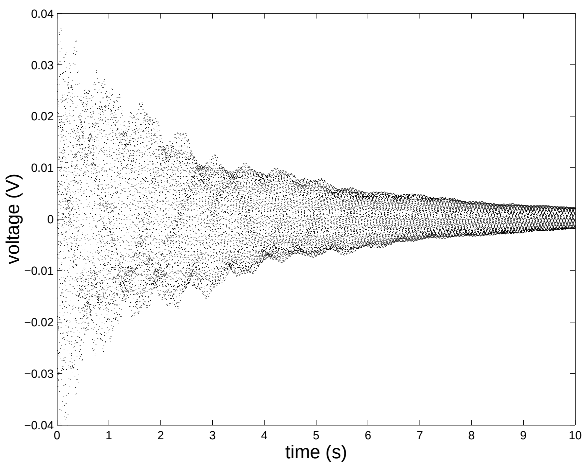

Figure 6.1. Pluck response with damper.

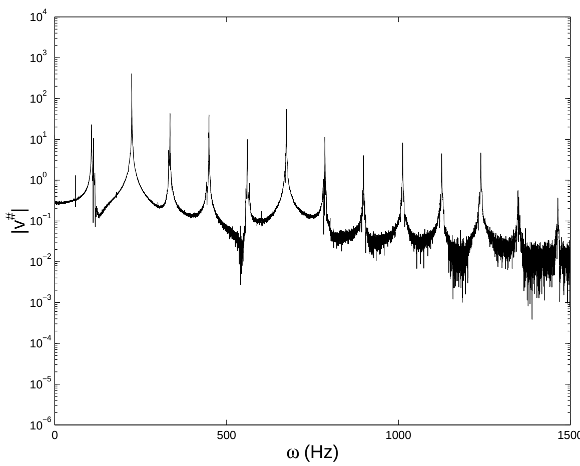

Figure 6.2. The magnitude of its DFT.

Comparing this to Figure 2.2 we note that the magnetic damper has indeed increased, or accelerated, the decay. To ascertain the associated natural frequencies we again turn, see Figure 6.2 to the Discrete Fourier Transform. Comparing the latter with Figure 2.3 we note that, although the resonant frequencies are essentially unchanged, the peak associated with the lowest frequency is severely diminished. That the lowest frequency is indeed the one that suffers the greatest dissipation is borne out by fitting the response of Figure 6.1 to the sum of damped sinusoids referred to as ϕ, recall (2.5). The p that solves (2.6) in this case is

$$p = \begin{pmatrix} -0.0059 & -1.3034 & 107.3376 & -1.6096 \\ -0.0228 & -0.2567 & 224.1213 & -1.4830 \\ -0.0048 & -0.4860 & 335.2361 & -1.0603 \\ -0.0027 & -0.2835 & 448.5797 & -2.6720 \\ 0.0012 & -0.5176 & 560.4651 & -5.5950 \\ -0.0074 & -0.6233 & 673.8908 & -7.3395 \\ -0.0017 & -0.7285 & 786.0627 & -10.1866 \\ -0.0008 & -0.9328 & 898.1001 & -10.7800 \end{pmatrix}$$

Notice that the decay, $p(1,2)$, of the lowest frequency, is 2 to 5 times greater than that of the next few. From this p we build the associated p^* and solve (5.8). Finding little improvement in the fit for $n > 8$ we plot below the best fit for $n = 8$.

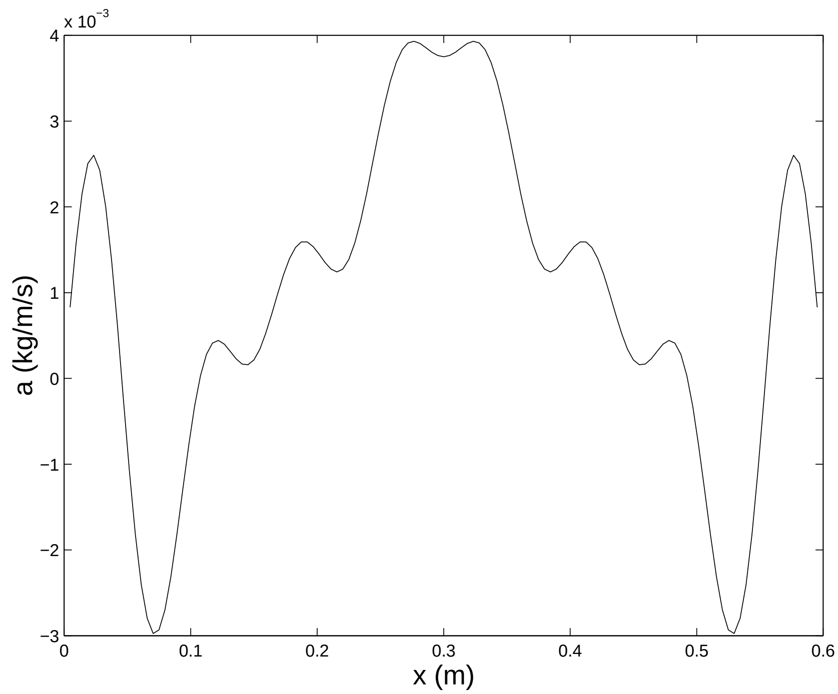

Figure 6.3. The magnetic rub.

Even with its ups and downs, this result is a strong indication that a dissipative device has been placed near the midpoint of the string.

7. Concluding Remarks

In my attempt at flushing out the rub I have invoked techniques and results from a number of fields. In the interest of both always keeping the goal in sight and actually producing an 'answer' you may accuse me of having taken giant leaps or, perhaps worse, having pursued my quarry without regard to rigor. Given the additional constraints of space I can only answer these criticisms by directing you to the relevant literature.

Regarding §2, even after more than 100 years, Rayleigh's text remains modern. It serves as a fascinating (and cheap) supplement to an introduction to partial differential equations. The engineering field of identifying dynamical systems by their modes of vibration is called Modal Analysis. Along these lines I highly recommend the text of Ewins [E].

For more on the history and application of the Principle of Least Action the best two sources are the lovely text of Hildebrandt and Tromba [HT] and the lecture of Feynman [FLS].

For a more careful derivation of the eigenvalues and eigenvectors of the undamped and damped wave operators I recommend the text of Weinberger [W]. You will also find there a study of strings that may undergo *longitudinal* as well as transverse motion.

To learn more about the approximation of operator eigenvalues by matrix eigenvalues, as practiced in §5, I would turn to the text of Chatelin [Ch].

And finally, you may ask, do the eigenvalues of the damped wave operator, $\mathcal{A}(a)$, indeed uniquely determine the damping a? We have offered here no more than numerical evidence in support of the conjecture. The proof, see Cox and Knobel [CK], relies on the clever implementation by Yamamoto of the so called Gelfand–Levitan transform. For the role of this transform in resolving questions of the type posed here I recommend the text of Levitan [L].

8. **References**

[A] Arnold, V.I., *Mathematical Methods of Classical Mechanics*, New York : Springer-Verlag, 1989.

[Ch] Chatelin, F., *Spectral Approximation of Linear Operators*, New York : Academic Press, 1983.

[CK] Cox, S.J. and Knobel, R. *An inverse spectral problem for a nonnormal first order differential operator. Integral Equations and Operator Theory, vol. 25, 1996, pp. 147–162.*

[E] Ewins, D.J., Modal testing : theory and practice, New York, Wiley, 1984.

[FLS] Feynman, R.P., Leighton, R.B. and Sands, M., The Feynman Lectures on Physics, Reading, Mass., Addison-Wesley Pub. Co. [1963-65]

[FR] Fletcher, N.H. and Rossing, T.D., The Physics of Musical Instruments, Springer, New York, 1991.

[GF] Gelfand I.M. and Fomin, S.V., Calculus of Variations, Prentice-Hall, Englewood-Cliffs, NJ, 1963.

[HT] Hildebrandt, S. and Tromba A., The Parsimonious Universe : Shape and Form in the Natural World, New York : Copernicus, 1996.

[L] Levitan, B.M., Inverse Sturm-Liouville problems, VSP, Zeist, 1987.

[R] Rayleigh, J.W.S, The Theory of Sound, Dover, New York, 1945.

[W] Weinberger, H. F., A First Course in Partial Differential Equations with Complex Variables and Transform Methods, New York, Blaisdell Pub., 1965.